THE SECOND AWARD OF THE BÔCHER MEMORIAL PRIZE

The Annual Meeting of 1924 was marked by the second award of the Bôcher Memorial Prize. On recommendation of the Committee, this prize, which was offered for a notable research memoir in volumes 19–24 of the TRANSACTIONS, was divided equally between Eric Temple Bell for a memoir entitled *Arithmetical paraphrases*, published in volume 22, and Solomon Lefschetz for a memoir entitled *On certain numerical invariants of algebraic varieties with applications to Abelian varieties*, in the same volume.

The presentation was made at the beginning of the afternoon session on December 30 by President Birkhoff as chairman of the committee. After some personal references to the mathematician whose name this prize commemorates, he gave the following brief resumés of the papers as analyzed by the committee.

"The memoir by Bell is a fundamental contribution to the theory of numbers. That field of mathematics is largely an aggregation of special results and methods. What have been needed are general principles, serving to unify and extend. Such a general principle is given by Bell. This principle may be stated very simply for a special but a typical case. Suppose that by the development of an elliptic or theta function we have found a linear identity between the sines of the angles $a_i x + b_i y$ for $i = 1, 2, \cdots$, then if $f(a, b)$ is an arbitrary even function, Bell shows that we have the like linear identity between the $f(a_i, b_i)$. Since we are free to choose the even function, we may deduce as many arithmetical facts as we please from the initial identity. Similarly from a linear identity between cosines we derive a corresponding identity for an arbitrary odd function. These are merely the simplest cases of the

endless varieties of paraphrases, each involving arbitrary functions.

"The memoir by Lefschetz constitutes an important contribution to the theory of algebraic varieties. In the first part of an extensive paper, rich in detail and employing with facility the methods of algebra, geometry, analysis, the notion of algebraic cycles plays a central part. It is shown for example that a double integral of the first kind on an algebraic surface has no periods relative to algebraic cycles, and that Picard's number ρ is equal to the number of two-cycles without periods, for integrals of the first kind. Here also is found the first rigorous proof of Noether's theorem on complete intersections, and the extension stating that a non-singular variety of d dimensions ($d>3$), which is itself a complete intersection in space of r dimensions ($r>4$) contains only hypersurfaces which are themselves complete intersections. The second part of the paper deals with Abelian varieties, and in particular it contains an evaluation of ρ for such varieties, known previously for the case $p=2$. Many interesting and novel results concerning the relations between the various numerical invariants are found."

Professor Lefschetz, who was present, in acknowledging the honor done him, spoke of the present standing of the American school of mathematics.

The Committee on award appointed by the Council consisted of Professor G. D. Birkhoff, L. E. Dickson and H. S. White.

The third award, to be announced at the Annual Meeting of 1928, is to be made with reference to the volumes of the TRANSACTIONS for the years 1923–1927 inclusive.*

R. G. D. RICHARDSON,
Secretary.

* See this BULLETIN, vol. 30 (1924), p. 193.

THE WESTERN CHRISTMAS MEETING OF THE SOCIETY

The twenty-second Western meeting of the Society was held on Friday, December 26, 1924, at the University of Chicago. The first session opened at 10 a. m. in Room 32, Ryerson Laboratory.

The attendance at this meeting numbered about sixty, including the following fifty-two members of the Society:

F. E. Allen, Barnard, Barnett, Bliss, O. E. Brown, Carmichael, Chapelon, H. H. Conwell, H. T. Davis, Dickson, Dostal, Dresden, Edington, Garabedian, Gouwens, G. H. Graves, V. G. Grove, Hildebrandt, Hodge, Hotelling, Louis Ingold, Ingraham, Isaacs, Kempner, Kinney, Kouperman, E. P. Lane, Latimer, Laves, Libman, March, G. A. Miller, Miser, E. H. Moore, E. J. Moulton, C. I. Palmer, Plapp, Rider, Roth, Rowland, Schottenfels, Shaw, Shohat, Simmons, Slaught, E. L. Thompson, J. S. Turner, Van Vleck, Vass, M. B. White, Frederick Wood, F. E. Wood.

The committee appointed to arrange for a symposium at the meeting in April 1925 reported that the symposium lecture, on "Some of the mathematical aspects of cosmology", would be given by Professor W. D. MacMillan.

At the dinner held on Friday evening, at the Del Prado Hotel, 34 persons were present.

Professor Hildebrandt, Vice-President of the Society, presided over the sessions, at which the following papers were presented. The papers of Professors MacNeish, Wahlin, Betz, Dodd and Ettlinger were read by title.

1. Professor F. E. Wood: *Sense relations between the pairs of corresponding triangles of a Desargues configuration.*

In this paper an investigation is made of the relative senses of the triangles in each of the ten pairs of perspective triangles of a Desargues configuration, starting with axioms and theorems developed in the foundations of geometry. A unique canonical lettering for a general Desargues configuration is set up and seven types of Desargues configurations with the canonical lettering are discussed. It is proved that the triangles of each of the ten pairs will have the same sense if and only if each

vertex is outside both the triangles corresponding to it, that the senses of the triangles of each of four and only four pairs are the same if and only if there is a vertex inside both its corresponding triangles, while if neither of the above two conditions is satisfied then the senses of the triangles of each of six and only six pairs are the same.

2. Professor E. P. Lane: *Bundles and pencils of nets on a surface.*

A bundle of nets on a surface is the class of nets such that the two component families of curves of each can be taken in such an order that at every surface point its tangents form with the tangents of a fundamental net the same cross ratio. The class of conjugate nets is a bundle for which the fundamental net is the asymptotic net and the constant cross ratio is -1. A pencil of nets in a bundle is a one-parameter family of nets in the bundle such that at every surface point the two tangents of each net form with the tangents of a fundamental net of the bundle a constant cross ratio.

This paper develops a general theory of bundles and pencils of nets. The theory finds application in a critical examination of the existing theory of conjugate nets; it shows what features of this theory are due to the constancy of the cross ratio and what features are due to the size of this constant. In these studies the importance of the ray-conic is again emphasized.

3. Professor V. G. Grove: *A theory of a general net on a surface.*

In this paper, the author sets up a complete system of invariants and covariants of the differential equations arising in the study of a general net. Necessary and sufficient conditions that a net be identically self dual are established. An extended study of Green's relation R between two congruences is made. There exists one and only one pair of congruences in relation R with respect to a general net, whose lines are at the same time reciprocal polars with respect to both quadrics osculating the parametric ruled surfaces of tangents. The associate conjugate net of a general net is discussed. It is shown that many of the properties of the associate conjugate net of a conjugate net hold when the net is general.

4. Mr. J. A. Shohat: *On the polynomial of the best approximation to a given continuous function.*

Consider two functions $f(x)$ and $p(x)$ defined on a finite interval (a, b); $f(x)$ is bounded and integrable; $p(x)$ is integrable and not negative. Assuming the existence of two numbers α, β such that $\int_{x_1}^{x_2} p(x)\,dx > 0$ $(\alpha \leqq x_1 < x_2 \leqq \beta;$ $a \leqq \alpha;$ $\beta \leqq b)$, the author proves the existence and the uniqueness of a polynomial $U_{n,k}(x)$ of degree $\leqq n$, minimizing the integral $\int_a^b p(x)\,|f(x) - U_{n,k}(x)|^k\,dx$, for any $k \geqq 1$.

Furthermore, if $f(x)$ is continuous and $\int_\alpha^\beta p(x)\,dx > 0$ for any α, β belonging to (a, b), then the said minimizing polynomial tends for $k \to \infty$ uniformly to the polynomial of the nth degree of the best approximation (in Tchebycheff's sense) in (a, b) to the function $f(x)$. Similar results hold for trigonometric approximation.

The results given above are a generalization of those obtained by G. Pólya (COMPTES RENDUS, 1913) and by D. Jackson (see the TRANSACTIONS and BULLETIN of the Society, 1921–1924).

5. Professor P. R. Rider: *The figuratrix in the calculus of variations.*

This paper considers integrals of the form $\int_{t_0}^{t_1} f(x, y, \tau) \sqrt{x'^2 + y'^2}\,dt$, $\int_{t_0}^{t_1} f(x, y, z, \tau, \sigma) \sqrt{x'^2 + y'^2 + z'^2}\,dt$, in which the arguments of f are functions of t. The quantities τ and σ are angles defined by the equations $\tan \tau = y'/x'$, $\sin \sigma = z'/\sqrt{x'^2 + y'^2 + z'^2}$. For the first integral, the figuratrix of the point (x, y) is defined as the envelope of the variable line $p \cos \tau + q \sin \tau = f(\tau)$, p and q being rectangular coordinates with reference to the point $P(x, y)$ as origin; for the second integral, the figuratrix of the point (x, y, z) is defined as the envelope of the two-parameter family of planes $p \cos \tau \cos \sigma + q \sin \tau \cos \sigma + r \sin \sigma = f(\tau, \sigma)$. The figuratrix as thus defined is proved to be identical with the figuratrix as defined by Hadamard (*Leçons sur le Calcul des Variations*, vol. I, pp. 92, 96). Attention is called to a number of the interesting properties of the figuratrix which enable one to interpret geometrically some of the well known functions and theorems of the calculus of variations. The results are generalized to the case of n dimensions.

6. Dr. H. F. MacNeish: *Theorems concerning transversals of the $(n+1)$-hedron in n-dimensional space.*

The line edges $A_i^0A_j^0 = a_{ij}$ of the $(n+1)$-hedron $A_1^0A_2^0\cdots A_{n+1}^0$ in n-space will be said to be divided antisymmetrically in the ratio $a_i:a_j$ by the point A_{ij}^0 if the ratio $A_i^0A_{ij}^0:A_{ij}^0A_j^0 = a_j:a_i$ where a_i, a_j, etc., are constants. In a triangle the 3 lines $A_i^0A_{jk}^0$ concur in the point A_{ijk}^0 which is said to divide the triangle antisymmetrically in the ratio $a_i:a_j:a_k$. The ratio $A_i^0A_{ijk}^0:A_{ijk}^0A_{jk} = a_j+a_k:a_i$ and $(A_i^0A_{jk}^0)^2 = (a_ja_{ik}^2 + a_ka_{ij}^2)/(a_j+a_k) - a_ja_ka_{jk}^2/(a_j+a_k)^2$. In a tetrahedron the following 7 lines concur in a point which is said to divide the tetrahedron antisymmetrically in a fixed ratio: (a) the 4 lines drawn from the vertices to the antisymmetric points of the opposite faces; (b) the 3 lines joining the antisymmetric points of the pairs of opposite edges. The three results (a) concurrence of transversals, (b) formulas for the lengths of the transversals, (c) the ratio of the segments of the transversals, stated above for 2 dimensions, are generalized to n dimensions.

7. Professor L. E. Dickson: *A new theory of the rational equivalence of linear transformations or pairs of bilinear forms.*

Let a linear transformation S on $\xi_1, \cdots, \xi_n$ with coefficients in any field F, the case of zero determinant not being excluded, replace any linear function x_1 of the ξ's by $x_1' = x_2$, x_2 by $x_2' = x_3, \cdots, x_{a-1}$ by $x_{a-1}' = x_a$, but x_a by a linear function $[x_1, \cdots, x_a]$ of those x's. Choose the leader x_1 so that a shall be the maximum length of all possible chains $x_1, \cdots, x_a$. If $n > a$, let b be the maximum length of a chain whose leader is linearly independent of $x_1, \cdots, x_a$. Continuing in this manner we may reduce S to a canonical form. The characteristic determinant of the partial transformation on any chain is divisible by that of the next chain. Apart from sign, these determinants are in reverse order the invariant factors, other than unity, of S. Two transformations S and T are similar in F (i. e., there exists a non-singular matrix B in F such that $BSB^{-1} = T$) if and only if the λ-matrices of S and T have the same invariant factors. Two pairs of n-rowed square matrices M, N and M_1, N_1 with elements in any field F, N and N_1 non-singular, are equivalent in F if and only if $M-\lambda N$ and $M_1-\lambda N_1$ have the same in-

variant factors. A like theorem follows when N and N_1 are singular, but the determinants of $\rho M+\sigma N$ and $\rho M_1+\sigma N_1$ are not zero identically in ρ and σ.

8. Professor Dickson: *Rational theory of pairs of bilinear forms in the singular case.*

The reduction of a pair of bilinear forms φ and ψ in the singular case in which the determinant of $u\varphi+v\psi$ is zero identically in u and v was first accomplished by Kronecker in the BERLINER SITZUNGSBERICHTE, 1890, pp. 1225–37. By a rational preliminary transformation he segregated component bilinear forms φ and ψ belonging to the non-singular case. For the latter, he employed the irrational canonical pair due to Weierstrass. Instead of these, we may employ the rational canonical pair which follows from the author's preceding paper. By purely rational transformation we may therefore reduce the initial φ and ψ to a canonical pair. The conclusion is that, in the singular case, two pairs of bilinear forms in the same $m+n$ variables with coefficients in any field F are equivalent in F if and only if they have the same invariant factors and the same minimal numbers, as defined by Kronecker.

9. Professor G. E. Wahlin: *On the solution of diophantine equations by means of ideals.*

This paper will appear in full in an early number of this BULLETIN.

10. Professor G. A. Miller: *Subgroup composed of the substitutions which omit a letter of a transitive group.*

The main object of this paper is to study transitive substitution groups G from the standpoint of the subgroups G_1 composed of all the substitutions which omit one letter. Among the theorems established are the following: The number of the transitive groups of degree $2k+1$ which have for their G_1 the group of order 2 and of degree $2k$ is the same as the number of the abstract abelian groups of order $2k+1$, k being an arbitrary positive integer. When G_1 is of order 2 and of degree 2^α the number of the possible degrees of the transitive groups which involve G_1 as the subgroup composed of all the substitutions which omit one letter is $\alpha+1$, where α is an arbitrary positive integer. A regular group of order

2^α, $\alpha>2$, cannot appear as the G_1 of a transitive group of degree $2^\alpha+2$. If a regular abelian group of order m is the G_1 of a transitive group of degree $m+k$ then $m\leqq 2k$.

11. Professor W. E. Edington: *The transform in the abstract definitions of groups.*

In this paper a study is made of some of the conditions in special cases imposed by the transform $s_2^{-1}s_1s_2 = s_1^\alpha s_2^\beta$ in the definition of a group whose generators are s_1 and s_2. Also the general relations that must hold between the orders of the generators and the orders of the products of powers of the generators when $s_2^{-1}s_1s_2 = s_1^\alpha$ are determined and the definitions of certain infinite systems of groups are discussed. Some interesting properties of the transform considered as a substitution are also stated.

12. Professor H. T. Davis: *Asymptotic distribution of characteristic numbers in the problem of the elastic bar.*

The problem of the elastic bar may be generalized by generalizing the Sturmian boundary value problem of fourth order to which it leads. The author has previously derived sufficient conditions that the characteristic numbers of this Sturmian problem shall alternate with the λ-zeros of a solution of a certain Cauchy problem of fifth order. In this paper a study is made of the Cauchy problem and the asymptotic form of the λ-zeros determined in terms of the coefficients of the Sturmian system.

13. Professor H. Betz: *Surface transformations applied to special dynamical problems.*

This paper applies to actual dynamical problems with two degrees of freedom, the surface transformation methods developed by Poincaré and Birkhoff. It has for its particular object the discovery of periodic motions, and the examination of their stability. The "ring of Poincaré" is constructed graphically affording an insight into the nature of the totality of motions. One periodic motion is then singled out for detailed investigation and extensive calculations are made in order to determine its stability.

14. Professor E. L. Dodd: *The frequency law of a function of several variables with given frequency laws.*

The functions considered are characterized by such pro-

perties as continuity and limited variation; and formulas are proved for effecting the required integrations, with special attention to justifying changes in the order of integration. The general theory is applied to determine the frequency law for the sum of projections and for the sum of cubes, under stated hypotheses regarding the frequency law for the arguments.

15. Professor L. Ingold: *Note on isogonal trajectories of geodesics.*

In this note conditions are determined on the function $a(u, v)$ in order that the curves $a = \text{const.}$ shall be isogonal trajectories of a family of geodesics $c = \text{const.}$ This special case of "parallel displacement", so called, is easily treated by vector methods. The resulting formulas are quite similar to the formulas of the more general case.

16. Professor I. A. Barnett: *The parameter group of a continuous group in function space leaving a manifold invariant.*

This paper extends Kowalewski's work on groups in function space to groups involving any number of functions in any number of variables. Three applications of such groups are given, viz., the parameter group of a given group, the adjoint group, and the parameter group of a given group leaving manifolds in function space invariant. Functional invariants and covariants of the manifolds and the forms defining them are also considered.

17. Professor H. J. Ettlinger: *Existence theorems for differential equations.*

In this paper the author proves the existence of solutions of a first order ordinary differential equation by methods analogous to the proof already given for the integrability of continuous functions. Superior and inferior integrals are obtained, which are analogous to the Darboux upper and lower integrals.

18. Professor A. J. Kempner: *Polynomials and their residue systems.*

In a paper bearing the same title (TRANSACTIONS OF THIS SOCIETY, vol. 22) the residue system of a polynomial of a single variable with integral rational coefficients with respect

to a modulus m (m a rational integer, prime or composite) was examined. The investigation is now extended to the case of a polynomial of more than one variable.

19. Dr. Harold Hotelling: *Economic problems involving maxima of functionals.*

The assumption is made that an entrepreneur will do all in his power to maximize an integral representing his future profits, discounted according to remoteness in time. A second assumption—usually unjustified in a capitalist society—that a machine will be operated to full capacity, reduces the problem of depreciation of replaceable assets to one in ordinary differential calculus. This is elaborated, with a refinement obtained by solving a Volterra equation, in a paper offered to the JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION.

The first assumption alone reduces many problems of economics to those involved in maximizing functionals, and in particular to problems of the calculus of variations. The very puzzling problem of mining economics reduces by an immediate first integration of the Euler equations to a system of first-order equations. This solves the problem of depreciation of irreplaceable assets, both under competition and under monopoly. This application is in fact broader than economics, since eudaemonistic ethical theories hold that an integral of future happiness is to be maximized, and so reduce right conduct to a problem in functionals.

20. Professor J. B. Shaw: *On certain linear algebras.*

This paper is concerned with the synthetic construction of algebras from their sub-algebras; particularly those that satisfy a relation $\xi = \Phi(\xi)\sigma$, where ξ is to be any and every number of the sub-algebra, Φ is a linear operator, constant as to ξ, dependent upon σ, and σ is a number not in the sub-algebra. An instance is Dickson's representation of Cayley's octave algebra in the form $q_1 + q_2$, where q_1 and q_2 are quaternions, and $q = \bar{q}\omega$. To be associative, $\Phi(\xi_1)\cdot\Phi(\xi_2) = \Phi(\xi_1\xi_2)$, that is, Φ must leave the sub-algebra automorphic in multiplication. The octave algebra mentioned is not associative. Examples are given from abstract groups considered as algebras.

ARNOLD DRESDEN,
Assistant Secretary.

THE THIRTY-FIRST ANNUAL MEETING OF THE SOCIETY

The thirty-first Annual Meeting of the American Mathematical Society was held at George Washington University, December 29–January 1, inclusive, in conjunction with the meetings of the Mathematical Association of America and the American Association for the Advancement of Science. Monday afternoon and Tuesday morning and afternoon were devoted to the regular scientific papers delivered before the Society. On Wednesday and Thursday mornings there were joint sessions with papers given by request; and on Wednesday and Thursday afternoons the Mathematical Association held its regular sessions. The meeting was brought to a conclusion by the dinner held jointly on Thursday evening with the Mathematical Association and Section A at the headquarters of the mathematicians, the Franklin Square Hotel. President Coolidge of the Mathematical Association presided at this function and some one hundred twenty-five were in attendance.

At the end of the Tuesday morning session it was voted to express to George Washington University and the department of mathematics the thanks of the Society for its cordial hospitality during this meeting, the largest in the history of the Society, and one of the most pleasant and most successful.

The attendance included the following one hundred fifty-five members of the Society:

C. R. Adams, O. S. Adams, R. B. Adams, Alexander, Archibald, G. N. Armstrong, Atchison, Bacon, Barney, W. J. Berry, Birkhoff, Bliss, Borden, E. W. Brown, H. S. Brown, Bruce, J. A. Bullard, W. G. Bullard, Cairns, Cajori, Caldwell, Carmichael, Chapelon, Chittenden, Clements, Clutz, Abraham Cohen, Coolidge, Currier, J. E. Davis, Denton, Dickinson, Dillingham, Dresden, Eiesland, English, Erwin, Everett, Feinler, Finkel, Fite, Foberg, Focke, Fry, Garretson, Gehman, Gerst, Glenn, Grant, Gravatt, L. M. Graves, Guggenbühl, Hancock, J. G. Hardy, G. A. Harter, Hawkesworth, Heal, E. R. Hedrick, Archibald Henderson, Robert Henderson, Hickson, Hildebrandt, Hille, Himwich, Hodgkins, Hughes, Hulbert, Hurwitz, Ingels, Karpinski, Kennelly, Kindle, Kline, W. D. Lambert,

Landry, Latimer, Lefschetz, Lehr, F. P. Lewis, Lunn, W. D. MacMillan, Manson, Mathewson, B. I. Miller, J. A. Miller, Mirick, H. H. Mitchell, C. N. Moore, R. L. Moore, Frank Morley, C. C. Morris, Marston Morse, F. R. Moulton, Murnaghan, F. H. Murray, Musselman, C. A. Nelson, Olds, Packer, Pell, F. W. Perkins, Plant, Rainich, Ramler, W. W. Rankin, Ranum, Rawlins, Remick, F. G. Reynolds, J. N. Rice, R. G. D. Richardson, H. L. Rietz, Ritt, Robert, G. E. Robinson, E. D. Roe, Roever, Root. Safford, J. B. Scarborough, Schoonmaker, Seely, Seidlin, Sharpe, Shenton, Siceloff, Silverman, T. McN. Simpson. Sinclair, Slaught, C. E. Smith, D. E. Smith, S. E. Smith, W. M. Smith, Synge, J. S. Taylor, J. M. Thomas, B. M. Turner, Tyler, Uhler, Van Orstrand, Veblen, J. L. Walsh, Wernicke, Whited, Widder, Wiener, Wiggin, A. H. Wilson, E. W. Wilson, Woodard, Woolard, Wyant, J. W. Young.

The following organizations were elected to Sustaining Membership:

Connecticut General Life Insurance Company;
Phoenix Mutual Life Insurance Company;
Sargent and Lundy, Construction Engineers.

The Council elected the following twenty-six persons to ordinary membership in the Society:

Mr. William Clarke Arnold, DePauw University;
Mr. Clyde Myron Cramlet, University of Washington;
Mr. Haskell Brooks Curry, Harvard University;
Mr. John H. Evans, Ohio National Life Insurance Company;
Miss Beatrice Alethea Fenner, Palo Alto;
Professor Robert Edwin Gaines, University of Richmond;
Mr. R. L. Grossnickle, University of Minnesota;
Mr. George Pryor Harmount, East High School, Columbus;
Professor Clyde Monroe Hobart, Northern Illinois State Teachers College;
Professor Theodore Jesse Hoover, Stanford University;
Mr. Luther William Hussey, Union College;
Mr. Arthur R. Jerbert, University of Washington;
Miss Myra I. Johnson, New York City;
Mr. Burton Wadsworth Jones, Western Reserve University;
Mr. John Joseph McCarthy, New York University;
Miss Muriel L. Metz, University of Cincinnati;
Professor Allan Benton Morton, Georgia School of Technology;
Professor William Pinkerton Ott, University of Alabama;
Mr. Paul E. Remington, Mountain States Telephone and Telegraph Company;
Dean Jonathan C. Rogers, Piedmont College;
Mr. Sergei Alexander Schelkunoff, Western Electric Company;
Professor Walter Francis Shenton, United States Naval Academy;
Mr. George W. Sublette, Minneapolis;
Mr. Walter Phillips Suesman, Providence;
Mr. William Herman Thurow, Technische Hochschule, Charlottenburg;
Miss Mabel Gertrude Whiting, Santa Ana Junior College.

The following twenty-six persons were elected to membership as nominees of sustaining members:

Mr. Horace Richardson Bassford, Metropolitan Life Insurance Company;
Mr. Lewis Warrington Chubb, Westinghouse Company;
Mr. Frank Conrad, Westinghouse Company;
Mr. Walter Dietz, Westinghouse Company;

Mr. George Myron Eaton, Westinghouse Company,
Mr. Robert D. Evans, Westinghouse Company;
Mr. Laurence R. Golladay, Westinghouse Company;
Mr. Clinton Richards Hanna, Westinghouse Company;
Mr. William J. Hudson, Westinghouse Company;
Mr. Robertson G. Hunter, Equitable Life Insurance Company of Iowa;
Mr. Philip C. Irwin, Equitable Life Insurance Company of Iowa;
Mr. Charles Jablow, Westinghouse Company;
Mr. Edwin W. Kopf, Metropolitan Life Insurance Company;
Mr. Reginald Campbell McCankie, Equitable Life Insurance Company of Iowa;
Mr. W. D. MacKinnon, Equitable Life Insurance Company of Iowa;
Mr. Samuel Milligan, Metropolitan Life Insurance Company;
Mr. Henry Scholte Nollen, Equitable Life Insurance Company of Iowa;
Mr. J. Ormondroyd, Westinghouse Company;
Mr. Robert James Piersol, Westinghouse Company;
Mr. Jonathan Gumersol Sharp, Metropolitan Life Insurance Company;
Mr. C. E. Skinner, Westinghouse Company;
Mr. Francis M. Smith, Metropolitan Life Insurance Company;
Mr. Carl Richard Soderberg, Westinghouse Company;
Mr. Hans Christian Specht, Westinghouse Company;
Mr. Dayton Ulrey, Westinghouse Company;
Mr. D. I. Vinogradoff, Westinghouse Company.

The ordinary membership in the Society is now 1477, including 94 nominees of sustaining members and 81 life members. There are also 19 sustaining members the total amount of whose annual dues is $3650. The total attendance of members at all meetings, including sectional meetings, during the past year was 509; the number of papers read was 231. The number of members attending at least one meeting was 365. At the annual election 210 votes were cast.

The reports of the Treasurer and the auditors (Mr. S. A. Joffe and Professor Tomlinson Fort) were received, showing a balance of $11,370.66, exclusive of special funds; of this, $5506.54 is reserved to secure the life memberships, and $2500.54 for publications for 1924 not yet completed. Sales of the Society's publications during the year amounted to $4938.58.

The Committee on Endowment announced that, including the pledges extending over five years, the Endowment Fund stands at approximately $60,000, of which $35,000 is paid in and invested.

The Board of Trustees adopted a budget for 1925,

showing estimated expenditures and receipts as $16,072 and $15,400 respectively; the deficit can be met from the cash surplus brought forward from earlier years.

The Librarian reported that the Library of the Society now contains 6733 volumes. This library has more than doubled in size since the last catalogue was published, in 1910; it is proposed to issue a new catalogue in 1925.

At the annual election, which closed on Tuesday morning, the following trustees and officers and other members of the Council were chosen:

Board of Trustees, Professor G. D. Birkhoff, Professor L. P. Eisenhart, Professor W. B. Fite, Mr. Robert Henderson, Professor R. G. D. Richardson.

President, Professor G. D. Birkhoff.

Vice-President, Professor G. C. Evans.

Assistant Secretary, Professor Arnold Dresden.

Member of the Editorial Committee of the Bulletin, Professor E. R. Hedrick.

Member of the Editorial Committee of the Transactions, Professor H. H. Mitchell.

Members of the Council, Dr. G. A. Campbell, Professors E. W. Chittenden, A. J. Kempner, H. E. Slaught, Virgil Snyder.

The tellers appointed by President Veblen to count the ballots were Professor J. F. Ritt, Dr. C. R. Adams, and Dr. Caroline Seely.

At the meeting of the Council, Professor L. P. Eisenhart was named to succeed himself as representative of the Society on the National Research Council. Professors Arnold Dresden and E. B. Stouffer were appointed to represent the Society on the Council of the American Association for the Advancement of Science.

It was voted to hold the Annual Meeting for 1925 in New York City, and that for 1926 in Philadelphia.

Statistics were presented by the Committee on Membership which show that of the approximately twenty-seven hundred teachers of mathematics in the colleges of the United States and Canada only one-third are members of the Society. In the past four years the membership of the Society has been doubled. A resolution was passed thanking Professor Clara E. Smith for her untiring efforts in connection with the membership campaign.

An account of the proceedings connected with the presentation of the Bôcher Memorial Prize appears on page 193 of this issue.

The second Josiah Willard Gibbs Lecture was delivered on Tuesday evening by Mr. Robert Henderson, vice-president of the Equitable Life Assurance Society. This lecture, which appears in full in the present number of this BULLETIN, was entitled *Life insurance as a social service and as a mathematical problem.* Professor James Pierpont has been invited to deliver the next lecture of this series.

At the joint session of the Society, the Mathematical Association of America, and Section A, held on Wednesday morning, the following papers were read:

I. *Remarks on the foundations of geometry,* by Professor Oswald Veblen, retiring President of the Society. This paper appeared in full in the March-April number of this BULLETIN.

II. *The foundations of the theory of algebraic numbers,* by Professor Harris Hancock, retiring Vice-President of Section A.

The program of the joint session held on Thursday morning with the Mathematical Association and Sections A, B, and D was as follows:

I. *Stellar evolution,* by Professor H. N. Russel. (Address delivered at the request of Section D.)

II. *Is the universe finite?* by Professor Archibald Henderson. (Address delivered at the request of the Mathematical Association of America.)

Titles and abstracts of the papers read at the regular sessions of the Society follow below. Retiring President Veblen presided at the sessions of Monday afternoon and Tuesday morning, relieved by Professor Coolidge. President Birkhoff presided at the Tuesday afternoon session. The papers of Adams (second paper), Bennett, Church, Dodd, Franklin, Graustein, Hazlett, Hurwitz (third paper), Kasner, Lubben, MacNeish, R. L. Moore (second paper), Ritt, Stone, Vandiver, Wiener (second paper), Wilder, and Zeldin were read by title; Mr. Merriman's paper was read by Professor C. N. Moore. Professor de la Vallée-Poussin was introduced by the Secretary.

1. Mr. H. S. Vandiver: *On methods for finding factors of large integers.* Second paper.

In the author's first paper with this title (this BULLETIN, vol. 30, p. 542) three new methods were described for factoring integers greater than eleven million. The present paper contains further details and examples in connection with the use of two of these methods. They are applied to the factorization of 13,179,643 and 236,364,091.

2. Mr. H. S. Vandiver: *Note on trinomial congruences and the first case of Fermat's last theorem.*

This note contains the proofs of some theorems concerning the relation

$$x^p + y^p + z^p = 0, \tag{1}$$

x, y and z being integers prime to the odd prime p, that depend on the possibility of finding prime integers q such that

$$\xi^p + \eta^p + \zeta^p \equiv 0 \pmod{q} \tag{2}$$

has no solution in integers ξ, η and ζ prime to q. In particular the following result is obtained: If (2) has no solutions under the conditions above mentioned, $q = 1 + mp$, and $m < 10p$, then (1) has no solutions in integers prime to p.

3. Mr. H. S. Vandiver: *Summary of results and proofs concerning Fermat's last theorem.*

This paper contains several results regarding Fermat's last theorem obtained by the author within the last ten years and not hitherto published. The proofs are indicated.

4. Mr. G. Y. Rainich: *Stereographic representation of surfaces.*

Given a fixed plane Π, a fixed point P outside Π and a surface Σ, we represent a point of Σ by the intersection with Π of the bisector of the angle formed at P by the perpendiculars to Π and to the tangent plane of Σ at the point represented. If we take cartesian coordinates of the representing point for parameters of the point represented, the surface can be determined by its second differential form; given the coefficients of this form, which must satisfy the Codazzi equations, we can find the parametric expression of the surface in the shape of three definite integrals. The Weierstrass representation of a minimal surface is a special case of this. The Codazzi equations being linear, we obtain a new solution by adding two known solutions; from the representations of a sphere and a minimal surface we obtain in this way a representation of surfaces of constant mean curvature. The method can be applied to hypersurfaces immersed in a flat space of next higher dimensionality. This is a particular case of the "modified tensor analysis" previously indicated by the author.

5. Mr. G. Y. Rainich: *On integrals in curved space.*

If a tensor field of rank $r+s$ in a flat space is alternating in r of its indices, an r-ple integral can be formed over an r-spread and this integral gives a tensor of rank s; but if the space in which the tensor field is considered is *curved*, the integration cannot be performed remaining in this space. In this case we choose s arbitrary vectors in the containing flat space, and form at each point of the curved space the inner product of the tensor field at this point and the projections on the corresponding tangent bundle of the s vectors of the containing space. In integrating the resulting tensor field of rank r over an r-spread in the curved space we obtain a multilinear function of s arbitrary vectors, i. e., a tensor of rank s, in the containing space. Conditions are discussed under which this tensor vanishes for a closed spread. Application is made of the foregoing in the cases of the indicator, the Riemann tensor, and the contracted Riemann tensor.

6. Miss Marguerite Lehr: *The plane quintic with five cusps.*

A modified form of del Pezzo's equation for this quintic suggests study of the system of conics through four cusps.

This method gives a parametric expression for the curve, involving only a quadratic irrationality, establishes the fact that the curve is unipartite, and that its five cusps and five inflexions occur alternately, and gives the equations of the inflexional tangents. The inflexions, inflexional tangents, cuspidal tangents and a number of points of interest on the curve are obtained from the five given cusps by linear constructions.

7. Professor Anna J. Pell: *A hyperbolic differential equation.*

It is shown by a special example that the spectrum associated with a hyperbolic differential equation with a boundary condition may be continuous.

8. Professor C. N. Moore: *On the application of Borel's method to the summation of Fourier's series.*

In the present paper it is shown that for the summation of Fourier's series Borel's method is less effective than Cesàro's. This is accomplished in two ways, first by a direct study of the kernel of the singular integral which arises from the application of Borel's method to such series, and second by exhibiting examples of continuous functions whose Fourier development cannot be summed by this method. On the other hand it is shown that while the Lebesgue constants corresponding to the application of Borel's method become infinite of the same order as the ordinary Lebesgue constants, it is not true, as one might expect, that the Borel method is no more effective than ordinary convergence in the study of Fourier's series. For it is possible to set up examples of continuous functions whose Fourier development is divergent at a certain point but which can nevertheless be summed there by Borel's method.

9. Mr. G. M. Merriman: *On a theorem concerning the convergence of triple series known to be summable* $(C1)$ *and its application to the triple Fourier series.*

In a paper presented to the Society, December, 1923, the author (1) proved a theorem deriving sufficient conditions for the convergence of double series from known results regarding their summability, and (2) applied this theorem to the double Fourier series, obtaining, as sufficient conditions for its convergence, generalizations of the Dini condition for the convergence of the simple Fourier series.

The present paper contains generalizations to three variables of both of these theorems: (1) a theorem giving sufficient conditions for the convergence of triple series summable ($C1$), and (2) an application of this theorem to obtain conditions for the convergence of a triple Fourier series. These latter are again generalizations of the Dini condition.

10. Dr. L. M. Graves: *The derivative as independent function in the calculus of variations.*

This paper contains some extensions of the results presented by the author under the same title at the meeting of the Society in Chicago, April, 1924. The integrals I considered are of the form

$$I = \int_{x_1}^{x_2} f\left(x, y_1 + \int_{x_1}^{x} z\,dx, z(x)\right) dx.$$

Admissible functions $z(x)$ are those which are bounded and integrable in the sense of Lebesgue. A necessary condition corresponding to that of Euler that an admissible function furnish a minimum for the integral I is deduced on the basis of very weak restrictions on the function f. Application of this theorem to the second variation yields a necessary condition corresponding to that of Jacobi for an admissible function to minimize I. Sufficient conditions are obtained that a function $z(x)$ continuous almost everywhere furnish a minimum for the integral I relative to the class of all admissible functions. The proofs are made without reference to any values of x outside the interval (x_1, x_2).

11. Dr. L. M. Graves: *On functionals defined implicitly.*

Lamson* has obtained a general implicit function theorem, but gives no information about the differentiability of the solutions of the functional equations he considers, which are of the two forms $y = F[y, z]$, and $G[y] = z$. In Part I of the present paper Lamson's classes of functions $[y]$ and $[z]$ are replaced by abstract sets concerning which the necessary properties are postulated, and conditions are given which ensure the existence of a solution $y = Y[z]$ of equations of the general type $G[y, z] = 0$ in the neighborhood of an initial solution (y_0, z_0). Further theorems are obtained concerning the existence of differentials and difference functions of all orders for the solutions $Y[z]$. In Part II a generali-

* AMERICAN JOURNAL, vol. 42, p. 243.

zation of the imbedding theorems for solutions of systems of ordinary differential equations is obtained from the theorems of Part I by considering sets whose elements are functions on a composite range. In Part III a number of special cases are discussed.

12. Dr. S. D. Zeldin: *The adjoint group in function space.*

In a number of papers published since 1911, mainly by G. Kowalewski, the Lie theory of continuous groups has been applied to function space. The author extends the notion of the *adjoint group* to problems in function space, and determines its relation to its isomorphic group.

13. Professor J. R. Musselman: *A configuration in space.*

Starting with a triangle in a plane, the author constructs a configuration of 18 points and 36 lines. Connected with it is a dual configuration of 36 points and 18 lines. The points fall naturally into 4 sets of three 3-points each; the whole is invariant under a ternary G_{216}, not the Hesse G_{216}. With a tetrahedron in space we are led to a configuration of 54 lines, 108 points and 108 planes such that 15 points are on each plane and 15 planes on each point. The lines fall into 4 sets of 27 each, such that on each set is a cubic surface of the Eckhardt type where the lines meet by threes in 18 points.

14. Professor H. F. MacNeish: *A nomogram in n-dimensional space for the solution of n linear simultaneous equations.*

In this paper a method is developed for solving n linear equations in n unknowns which is a generalization of the nomogram in the plane for the solution of 2 linear equations in 2 unknowns. This generalization is accomplished by the method of antisymmetric ratios considered in a paper *Theorems concerning transversals of the* $(n+1)$*-hedron in n-dimensional space* previously presented to the Society by the author.

15. Professor W. C. Graustein: *Semi-parallel maps of surfaces.*

If a map of two surfaces is called "parallel" it is assumed, in general, that there exist two systems of curves, one on each surface, having the property that their curves correspond and have at corresponding points parallel tangents.

A map is given the name "semi-parallel" if there exist merely two families of curves, one on each surface, which have this property. The general semi-parallel map and important special cases are discussed in this paper.

16. Professor W. C. Graustein: *An invariant of a general transformation of surfaces.*

If two surfaces are in one-to-one correspondence, there exists between the pencils of tangent lines at corresponding points a projective correspondence. The two pencils cut the line of intersection of their planes in two projective ranges of points. The invariant cross ratio of this projectivity is introduced as an invariant of the transformation of the one surface into the other. Applications of it are made to the so-called fundamental or F transformations.

17. Professor O. E. Glenn: *On the residuation of systems of binary forms.*

The two-dimensional system Q_1 of integers generated from an infinite series $S(\alpha)$: $\alpha_1, \alpha_2, \cdots$, as first column, in which any element is $e = x\gamma + y\delta$, where δ is the element immediately preceding e in the row and γ the one next beneath δ in the column, has, for its general element, the binary form $(\alpha_{v-u+1}, \alpha_{v-u+2}, \cdots, \alpha_v)(x, y)^{u-1}$. This paper is a study of the residue system Q of Q_1, modulo n. Subjects treated are generation of periodic sequences $S(\alpha)$ by means of periodic functions, period parallelograms and drift lines, prime elements. Some of the periodicities considered are manifestations of Fermat's theorem, Euler's function. $\varphi(n)$, and the theory of exponents.

18. Mr. C. G. Latimer: *The arithmetics of generalized quaternion algebras.*

The author considers the algebra over the field of rational numbers with the units $e_0 = 1$, e_1, e_2, e_3 and the following multiplication table: $e_i e_0 = e_0 e_i = e_i$ $(i = 0, 1, 2, 3)$, $e_1^2 = -\alpha$, $e_2^2 = -\beta$, $e_3^2 = \alpha\beta$, $e_1 e_2 = -e_2 e_1 = e_3$, $e_1 e_3 = -e_3 e_1 = \alpha e_2$, $e_2 e_3 = -e_3 e_2 = \beta e_1$, where α and β are integers $\neq 0$. He finds systems of integral elements, as defined by Dickson*, when α and β are odd and also in one case when α is even and β odd.

* See his *Algebras and their Arithmetics*, § 87.

19. Professor F. R. Sharpe: *The irrationality of the general cubic variety in four-dimensional space.*

It has long been considered probable but it has not been definitely proved that the general cubic variety in four-dimensional space is irrational. In the present paper rational cubic varieties are studied by means of the space involution derivable from the hyperplane sections through a point on the variety, and the conclusion is drawn that in all cases the variety is not the general cubic variety. If this is the case, then the general cubic variety must be irrational.

20. Professor Edward Kasner: *Analytic curves for which the chord equals the arc.*

The problem is to find regular analytic curves through a given point O such that the arc OP equals the chord OP for every point P. In complex space there is a large class of such curves besides the straight lines, which are the only real solutions. In an ordinary three-space the curves lie in minimal planes. Generalization to Riemann space of n dimensions is discussed.

21. Professor J. L. Synge: *A generalization of the Riemannian line-element.*

This paper outlines the geometry that results from the assumption that the square of the line-element is a homogeneous function of the second degree of the differentials of the coordinates, without the additional hypothesis of Riemann that this function is quadratic. It is found that the equations of the geodesics may be presented in their usual form,

$$\frac{d^2x^i}{ds^2} + \begin{Bmatrix} jk \\ i \end{Bmatrix} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

where the Christoffel symbols contain the coordinates and are homogeneous functions of zero degree in the first derivatives of the coordinates with respect to s. The natural generalization of the Levi-Civita formula for parallel propagation of a vector is found to contain the second derivatives of the coordinates with respect to the parameter as well as their first derivatives. The "angle between two vectors with respect to a given curve" and other geometric concepts are discussed.

22. Professor G. A. Bliss: *Boundary value problems for systems of linear differential equations of the first order.*

In this paper the author studies the problem of determining the solutions of a system of differential equations

$$\frac{du_i}{dx} = \sum_{k=1}^{n}[A_{ik}(x) + \lambda B_{ik}(x)]u_k(x) + g_i(x), \quad (i = 1, \cdots, n),$$

with boundary conditions of the form

$$s_i(u) = \sum_{k=1}^{n}[M_{ik}u_k(a) + N_{ik}u_k(b)] = h_i, \quad (i = 1, \cdots, n),$$

where the coefficients M_{ik}, N_{ik} and h_i are constants. A very large number of boundary value problems for ordinary differential equations, hitherto separately studied in the literature, are expressible as special cases of this. It is shown that when the problem is definitely self-adjoint according to a suitable definition the characteristic numbers λ_r are denumerably infinite in number and real, and each has associated with it a unique set of characteristic functions $u_{ir}(x)$ $(i = 1, \cdots, n)$. Every set of functions $f_i(x)$ $(i = 1, \cdots, n)$ with suitable continuity properties which satisfy the boundary conditions $s_i(f) = 0$ $(i = 1, \cdots, n)$ is expressible uniquely in the form $f_i(x) = \sum_{r=1}^{\infty} c_r u_{ir}(x)$ $(i = 1, \cdots, n)$, and the series converges uniformly on the interval $a \leqq x \leqq b$.

23. Professor W. A. Hurwitz: *On de la Vallée Poussin summability.*

The definition of summability given by de la Vallée Poussin has been studied by Plancherel, C. N. Moore, Gronwall, and Kogbetlianz. In the present paper, the author shows its equivalence to other similar definitions, generalizing results of Kogbetlianz, proves that it permits omission and adjunction of terms, establishes an appropriate necessary condition for summability, and considers the relation of the definition to other standard methods of summability.

24. Professor W. A. Hurwitz: *Characteristic parameter values for an integral equation.*

It is well known that the characteristic parameter values for a real symmetric (more generally, complex hermitian) kernel are all real; this result is easily extended to give conditions that the characteristic values lie on any straight

line through the origin. In this paper, the author gives like criteria for a straight line not through the origin. A typical result is the following: If $\alpha \neq 0$, and if

$$\bar{\alpha} K(x, y) + \alpha \bar{K}(y, x) = 2 \int K(x, s) \bar{K}(y, s) ds$$

$$= 2 \int \bar{K}(s, x) K(s, y) ds$$

the characteristic parameter values λ for the kernel $K(x, y)$ lie on the straight line $R(\lambda\alpha) = 1$.

25. Professor W. A. Hurwitz: *A trivial Tauberian theorem.*

An example is given of a Tauberian theorem, otherwise unimportant, which exhibits cases of each of the customary Tauberian conditions ($|u_n| < Kf_n$, $u_n < Kf_n$, $u_n > -Kf_n$, $u_n/f_n \to 0$), and thus throws light on the occurrence of the different types.

26. Professor Marston Morse: *Relations "in the large" between the numbers of extremals of different types joining two fixed points.*

In this paper the author is concerned with the "regular" plane problem in the non-parametric form. He shows that under certain boundary conditions the problem of the determination of the extremals joining two fixed points is equivalent to the problem of determining the critical points of a function, f, of a large but fixed number of variables. On any extremal joining the two given points the number of points conjugate to either end point determines the nature of the corresponding critical point of f. In a paper presented to the Society in December, 1923, the author gave a set of relations between the numbers of critical points of different types of a function f under certain boundary conditions. These relations lead at once to a set of relations between the numbers of extremals of the different types joining the two fixed points.

27. Dr. C. R. Adams: *An existence theorem for a linear partial difference equation of the intermediate type.*

The equation considered is $g(x+1, ry) = a(x, y) g(x, y)$, in which r is a constant, real or complex, not zero and

in absolute value different from unity; $a(x, y)$ is a given polynomial in x and y (no added generality is secured by taking it a rational function); and $g(x, y)$ is the function to be determined. There exist in general two formal series solutions, the first in powers of $1/x$ and y and the second in powers of $1/x$ and $1/y$. These series do not in general converge. There exist, however, two solutions which are analytic save for poles over the entire x- and y-planes ($y = 0$ excluded for the second solution) and which, in a certain sense, are asymptotically represented by the corresponding formal series. Roughly speaking, the sense is this: the first solution is represented by the first formal series with respect to x in any half-plane and with respect to y in the complete neighborhood of the origin; the second is represented asymptotically by the second formal series with respect to x in any right half-plane and with respect to y in the complete neighborhood of infinity.

28. Dr. C. R. Adams: *Note on the existence of analytic solutions of non-homogeneous linear q-difference equations, ordinary and partial.*

This paper deals with a system $g_i(qx) = \sum_{j=1}^{n} a_{ij}(x)g_j(x)$ $(i = 1, 2, \cdots, n-1)$, $g_n(qx) = \sum_{j=1}^{n} a_{nj}(x)g_j(x) + b(x)$, no loss in generality being suffered in taking only one equation of the system to be non-homogeneous. If the coefficient functions $a_{ij}(x)$ and $b(x)$ are rational and if $|q|$ is $\neq 1$, the system in general possesses two formal series solutions. These series converge in a large class of cases and so represent, in certain regions, analytic solutions. The region of analyticity of the solutions can be extended by repeated use of the system itself. The relation between the two solutions is found to be expressible in terms of solutions of the associated homogeneous system (obtained by deleting $b(x)$) and certain periodic functions the nature of which can, in important instances, be completely determined. Similar theorems are proved for a system analogous to the above but in which the functions involved depend on more than one variable*.

* More precisely, a non-homogeneous system of the class treated by the author in a recent paper *The general theory of a class of linear partial q-difference equations*, TRANSACTIONS OF THIS SOCIETY, vol. 26 (1924), pp. 283–312.

29. Professor E. R. Hedrick: *A generalization of Morera's theorem.*

Generalizations of Morera's theorem have been stated by several authors. According to J. Wolff, who gave one such generalization in a paper entitled *On sufficient conditions for analyticity* at the Toronto Congress, Looman has given generalizations in papers in NIEUW ARCHIEF VOOR WISKUNDE (vol. 14, pp. 234–239). The present paper gives still further generalizations. Let us first assume that the real and the imaginary parts of a function of a complex variable have continuous first derivatives, and let us employ the expression *analytic at a point* in the sense defined by the author and others (LIOUVILLE, (9), vol. 2 (1923), pp. 327–342). Then it is shown in the present paper that a necessary and sufficient condition that a function be analytic at a point is that the limit of the ratio of $f(z)dz$ over a square about that point to the area of the square approach zero with the area, on a given sequence of squares. It follows that a necessary and sufficient condition that the function be analytic at every point is that the same property hold at each point of a set everywhere dense in the region.

30. Professor J. F. Ritt: *A theorem on rational functions analogous to a theorem of Lüroth.*

If $F(z) = \varphi_1[\varphi_2(z)]$, all three functions being rational, $\varphi_2(z)$ will be called a forefactor, and $\varphi_1(z)$ a postfactor of $F(z)$. It is a well known theorem of Lüroth that given any two rational functions, $F_1(z)$ and $F_2(z)$, there exists a rational combination $\varphi(z) = R(F_1, F_2)$ of the two functions, such that $F_1(z)$ and $F_2(z)$ are both rational functions of $\varphi(z)$. This theorem implies immediately, and is almost equivalent to, the following result: Any two rational functions, $F_1(z)$ and $F_2(z)$, have a forefactor $\varphi(z)$ in common such that every common forefactor of $F_1(z)$ and $F_2(z)$ is a forefactor of $\varphi(z)$. The present paper investigates the common postfactors of a pair of functions. A result analogous to the above is obtained, namely that any two rational functions have a common postfactor of which every common postfactor is a postfactor. The writer's proof of this theorem does not resemble that of the theorem of Lüroth. The dissimilarity seems actually to be due to differences in the nature of the theorems.

31. Professor Olive C. Hazlett: *Formal modular covariants as algebraic invariants.*

In this paper it is shown how, as a result of a symbolic theory developed several years ago, it follows that any isobaric formal modular invariant of a binary form f with respect to the Galois field, $GF[p^n]$, of order p^n is congruent, mod p, to an algebraic invariant A of the form f, where A is a rational integral invariant if $p \neq 2$ and is either a rational integral invariant or a square of such if $p = 2$. Another type of formal modular invariant of $f(a; x, y)$ is an algebraic invariant of $f(a^{p^n}; x, y)$ and $f(a; x, y)$. Similarly, the formal modular invariants of $f(a; x, y)$ are in general algebraic invariants of $f(a; x, y)$ and certain related forms. By applying these results, the author attempts to determine a general method of finding the formal modular invariants and covariants of a general form from algebraic invariants, which shall be of general applicability.

32. Professor C. de la Vallée Poussin: *On functions differentiable an infinite number of times.*

This paper discusses (1) general relations between the law of limitation of successive derivatives and that of the Fourier coefficients of a periodic function, and (2) the existence of functions whose successive derivatives take given values at a point.

33. Professor R. L. Moore: *Covering theorems.*

Using the following definition: A point set K is said to be *uniformly relatively well imbedded* in a set of regions G if there exists a positive number d, less than unity, such that each point of K belongs to some region R of G and is at a distance from the boundary of R less than d times the diameter of R, the author proves these theorems: (1) If, for each circle g of a set G, h_g is a definite circle concentric with g and there exists a positive e, less than unity, such that, for every g, the radius of h_g is less than e times that of g, then the set G contains a subset such that, for each g, h_g is within one and only one circle of this subset. (2) If G is a set of circular regions and K is a point set of measure zero, and, for every positive e, K is uniformly relatively well imbedded in a subset of G whose regions are of radius less than e, then, for every positive ϵ, G contains a subset G^* which covers K, the sum of the areas

of the regions of G^* being less than ϵ. (3) J. Splawa-Neyman's theorem (FUNDAMENTA MATHEMATICAE, vol. 6) remains true if the words "à son intérieur" are omitted.

34. Professor R. L. Moore: *Concerning the separation of a continuum which does not separate the plane.*

A subset H of a point set K is said to separate the point A from the point B in K in the weak sense if $K-H$ contains no continuum which contains both A and B. In this paper it is shown that if, in a plane S, A and B are two distinct points of a bounded continuum K no subset of which separates S, and G is a collection of subcontinua no one of which separates A from B, in K, in the weak sense, then, even if there be uncountably many continua in the collection G, their sum cannot separate A from B, in K, in the weak sense.

35. Professor R. L. Moore: *Concerning upper semi-continuous collections.*

Suppose G is an upper semi-continuous collection of mutually exclusive continua which fill up a plane S. In this paper it is shown that, if the continua of the set G are considered as ideal points and the term region is suitably defined, the set of all such ideal points and regions satisfies the set of axioms $\sum_1$ of the author's paper *On the foundations of plane analysis situs,* and thus has all the topological properties of an ordinary plane. Thus we have a theory of arcs, simple closed curves, continuous curves, etc., whose elements are continua of the set G. For example, a simple closed curve of continua of the set G is a closed and bounded set of such continua which is disconnected by the omission of any two continua of G which belong to it.

36. Mr. H. M. Gehman: *A characterization of simple continuous curves.*

It is proved that if M is a closed and connected set of points in a euclidean plane, such that $M-W$ is uniformly connected im kleinen relative to W, where W is any closed and connected subset of M consisting of more than a single point, then M is a simple continuous arc, a simple closed curve, a ray, or an open curve.

37. Professor Norbert Wiener: *Quasi-analytic functions and quadratic forms.*

The author develops an extension of the Denjoy-Carleman theory of quasi-analytic functions, making use of the properties of a certain quadratic form in the Fourier coefficients of a function.

38. Professor Norbert Wiener: *The representation of functions by trigonometric integrals.* Part 2.

The author develops a method for deducing Bohr's relation between the sum of the squares of the coefficients of a quasi-periodic function and the average of the square of the function from the general theory of the trigonometric expansions of bounded functions.

39. Professor Norbert Wiener: *Note on a paper of O. Perron.*

The author points out that the methods of solving the Dirichlet problem in its generalized sense, given by O. Perron in a recent paper in the MATHEMATISCHE ZEITSCHRIFT, enable one to build up a generalization of the Dirichlet problem entirely equivalent to that exhibited by the author in two recent papers.

40. Dr. F. H. Murray: *On certain families of orbits with arbitrary masses in the problem of three bodies.* Second paper.

In this paper a discussion is given of the stability of the straight line and equilateral triangle solutions, respectively, in the problem of three bodies, by means of the results of two preceding papers. It is shown that certain generalized theorems of Bohl can be applied directly to a neighborhood of the straight line solutions, with arbitrary masses, and to a neighborhood of the equilateral triangle solutions, if the masses are such that the characteristic exponents of the generating orbits are not all pure imaginaries; the mutual distances are assumed constant for the generating orbits in both cases.

41. Mr. F. W. Perkins: *On the oscillation of harmonic functions.*

Given any circle C and a function u finite and harmonic within C, let the oscillation of u on C (denoted by $Du|c$) be defined as the difference between the upper and lower

bounds of u within C. The most important of the new results in the present paper are contained in the following theorems: Let P_1, P_2 be any two points within a circle C, and draw the two circles C_1, C_2 through P_1 and P_2 which are tangent to C. Denote by A_1, A_2 the angles subtended in these circles by the chord P_1P_2, the vertices being at the points of tangency of C with C_1, C_2 respectively. Let u be any function which is finite and harmonic within C. Then $|u(P_1)-u(P_2)| \leqq [(A_1+A_2)/\pi]Du|c$. Let O be the center and A the radius of a circle C, and let P be a point interior to C, at a distance ρ from 0. Let u be any function finite and harmonic within C, and denote by $[du/d\xi]_P$ the directional derivative of u at P in an arbitrary direction ξ. Then $|[du/d\xi]_P| \leqq (2/\pi)[A/(A^2-\rho^2)]Du|c$.

42. Mr. R. M. Foster: *The driving-point impedance of an electric circuit.*

Necessary and sufficient conditions that a rational function $S(\lambda)$ represent the driving-point impedance of an electric circuit consisting of resistances, inductances, and capacities for oscillations proportional to $e^{\lambda t}$ are determined in this paper for all cases in which the impedance function $S(\lambda)$ is of the first or second degree, by determining (1) the domain of values within which the coefficients of S must lie, (2) the corresponding domain of values for the roots and poles of S, and (3) the form of the S curve for real sinusoidal oscillations, that is, for $\lambda = 2\pi fi$, the frequency f increasing from zero to infinity.

43. Professor J. S. Taylor: *A note on the theory of depreciation.*

In a paper previously presented to the Society, the author attacked the problem of distributing the depreciation charges for a machine having a measurable unit output so that the unit cost of production for an average machine would remain constant (or vary in any preassigned way) during the period of use, and of determinig the period of use so that this unit cost would be a minimum. In the present paper, the problem is that of determining the period of use and distributing the depreciation charges so that the profit shall be a maximum. As before, the method is dependent upon the availability of statistical information concerning the behavior of the type of machine

in question. Particular attention is called to the dependence of the useful life upon the market value of used machines, with the attendant question of the relative desirability of the purchase of second hand machines instead of new ones.

44. Professor E. W. Chittenden: *A simple proof of the topologic invariance of the sets $F_\sigma\delta$ of Hausdorff.*

The sets $F_\sigma\delta$ of Hausdorff are products of sums of closed sets. S. Masurkiewicz has proved that every set which is homeomorphic with a set $F_\sigma\delta$ is also a set $F_\sigma\delta$. It is the purpose of the present note to present comparatively simple proofs of this proposition and a related theorem of Sierpinski. This theorem states that the property of being the greatest limit of a sequence of closed sets is also a topologic invariant.

45. Professor A. A Bennett: *Generalized elementary symmetric functions.*

The plan proposed in a previous paper by the author is here carried out. A study is made of the general linear function in a domain of integrity. By combinatorial methods alone, the usual properties of the elementary symmetric functions are established in this more general case. Thus, in particular, Newton's identities, Waring's formula, and related theorems are proved. By introducing further conditions such as the finite existence of $f(1)$, where f is the linear function, a general norm is defined and the usual properties of the determinant are proved in so far as these are independent of the structure of an assumed matrix. Thus all results are equally valid for the determinant of a linear transformation and the norm of an algebraic number. Incidentally, totally new proofs of Newton's identies and related theorems are given which are elementary although frequently not as compact as the usual discussions. The entire treatment is necessarily independent of any appeal to the conjugates of a given element.

46. Professor A. A. Bennett: *Semi-groups.*

The notion of semi-group, or of a set with the "associative group property" is fundamental. Recent widespread interest in linear algebras has developed many concepts foreign to the classical group theory but belonging properly to semi-groups. The author examines the semi-group on

its own account. Important concepts are the idempotent elements and absorbent elements. Thus a absorbs b if $ab = ba = a$. Among the theorems proved are the following: In each semi-group there exists a sequence $a_1, a_2, \cdots, a_n$, where each a_i absorbs a_{i+1}, and where the sequence cannot be extended by inserting new terms or in other prescribed ways. In such a complete sequence each non-idempotent element is either immediately preceded or immediately succeeded by an idempotent element. If A and B are semi-groups, and A is a subset of B, such that $AB = BA = A$, A is said to be a *normal* subset of B. It seems possible to accomplish almost or quite as much with this concept of normal subset as with that of invariant subset, while numerous special theorems are shown to hold for normal subsets which are not applicable to the less specific concept of invariant subset.

47. Professor A. A. Bennett: *Five axioms for point and translation in affine geometry.*

This paper appeared in full in the November-December number of this BULLETIN.

48. Professor A. A. Bennett: *Biaffine geometry.*

In this paper the author introduces biaffine geometry as a subject of study on its own account. Geometrically, it may be described as the affine geometry of vectors, and analytically it is the study of homogeneous non-singular linear transformations, without the use of arbitrary multipliers. This geometry enjoys the same duality manifested by projective geometry, while remaining finite. Thus there is parallelism among points as among lines. Parallel coplanar lines fall into two mutually exclusive separated sets. The vector has a dual, the "sector", analogous to an angle in some respects. Conics fall into many types, some self-dual, others not. Points may be added and subtracted except under certain exceptional restrictions. The present study covers points, lines, and conics only.

49. Professor R. L. Wilder: *On connected sets which cut the plane.*

Let M be a connected set (containing more than one point) which lies in a plane S and consists entirely of non-cut points but which is disconnected by the omission

of any two of its points. Under these conditions the author shows that M cuts the plane S in the sense that there exist at least two points, A and B, of $S-M$, which do not lie in a subcontinuum of $S-M$. If to the above conditions be added the condition that M is closed, M is, in fact, a simple closed (Jordan) curve.

50. Mr. R. G. Lubben: *On the approximate covering of a point set by a finite set of chains.*

The author shows that if, in space of one dimension, G is a set of intervals covering a point set K, then, for every positive number e, K contains a subset K_e, of exterior measure less than e, such that $K-K_e$ is covered by a finite number of non-overlapping finite simple chains whose links are intervals of the set G.

51. Professor E. L. Dodd: *The frequency law of a function of one variable.*

This paper has appeared in full in the January-February issue of this BULLETIN.

52. Mr. M. H. Stone: *A comparison of the series of Fourier and Birkhoff.*

This paper consists of a study of comparative properties of Fourier's series and of the series defined by Birkhoff in volume 9 of the TRANSACTIONS OF THIS SOCIETY, with respect to any summable function, and an investigation of the term-by-term derived series of Birkhoff's series.

53. Mr. M. H. Stone: *Irregular differential systems of order two and the related expansion problems.*

The problem of the irregular expansions arising from differential systems of order two, as described by Birkhoff in volume 9 of the TRANSACTIONS OF THIS SOCIETY, is here discussed by methods which are an extension of those employed in the paper cited.

54. Dr. Philip Franklin: *Weierstrass's approximation theorem.*

Lebesgue has given a simple proof of the Weierstrass approximation theorem for continuous functions of one variable, but the extensions to more variables involve

somewhat more complexity. In this note a simple proof is given for the functions of several variables.

55. Mr. Alonzo Church: *On irredundant sets of postulates.*

A set of postulates is irredundant if the postulates are independent and no one of them can be replaced in the set by a weaker postulate in such a way that none of the implications of the set are lost. A necessary and sufficient condition that a set of postulates be irredundant is that the negatives of every two be contradictory. This paper contains an irredundant set of postulates for the system of positive and negative integers, and this set is shown to be categorical. The general problem is proposed to find irredundant sets of postulates of not too complicated form.

56. Professor Edward Kasner: *Null surfaces.*

The author means by a null surface one whose ds^2 vanishes identically. Such surfaces do not exist in three-space, but do exist in space of four or more dimensions. Geometry in a null plane or surface must be distinguished from geometry in a minimal plane or developable. Null m-spreads in a given n-spread are discussed.

R. G. D. RICHARDSON,
Secretary.

A CORRECTION

BY D. N. LEHMER

On page 401 of volume 8 of this BULLETIN (June, 1902), there is a list of errors in Legendre's tables of linear divisors of quadratic forms. The correction stated under the heading III is wrong. The form $172\,x+147$ is correct for the form t^2-43u^2.

LIFE INSURANCE AS A SOCIAL SERVICE AND AS A MATHEMATICAL PROBLEM*

BY ROBERT HENDERSON

The prophet Ezekiel in describing the wonderful vision which appeared to him by the river Chebar uses an expression which has become proverbial. He describes four living creatures and speaks of each as accompanied by a wheel. Regarding these wheels he says "their appearance and their work was, as it were, a wheel within a wheel". I hope that if you get from this talk of mine the impression that the business of life insurance is like the vision of Ezekiel "wheels within wheels", I shall be able to make clear to you that it is like that vision also in the further respect that as Ezekiel also says "the spirit of life was in the wheels". If there is, or ever was, a business which is the embodiment and outgrowth of a vital principle it is that of life insurance.

That life insurance performs a very important social service is shown by the remarkable vitality through the ages of institutions attempting in a small way to furnish benefits which were essentially life insurance and by its remarkable growth in modern times after the correct scientific basis had been discovered.

The recorded history of the world has largely to do with states, their rulers and their wars, and it is only by the chance survival of documents that we learn of those social forces and institutions which did not come to the attention of the rulers as subjects of regulation. It is probably for this reason that we first hear of insurance benefits in connection with voluntary associations or collegia because, while voluntary associations appear to have been comparatively free in the days of the Roman republic they were

* Delivered before the Society and their guests, as the Josiah Willard Gibbs Lecture, at Washington D. C., December 30, 1924.

the subjects of very rigid regulation under the emperors and only those whose objects met with the approval of those in authority were permitted to exist legally.

The particular associations which come to our attention as embodying life insurance benefits appear to have been in the first place religious societies devoted to the worship of particular deities. In view of the religious element involved in funeral rites a provision for the performance of those rites at the death of members of the society was, therefore, a natural development and might be considered as merely incidental were it not for the fact that this feature appears to have had such vitality that associations came into existence which, so far as we know, had no other object. Some of these societies had their own burial place or columbarium, according to the mode adopted for disposing of the body, while others seem to have merely paid a cash sum at death for the purpose of defraying the necessary expenses.

The membership of these societies of which we are speaking was composed of slaves and other very poor people, so that the benefits were extremely small. For example, in one case the entrance fee was equivalent to three dollars and a half and an amphora of good wine and the monthly contribution to about four and one-half cents. The funeral benefit paid by this society was equivalent to about ten and one-half dollars.

These societies, called Collegia Tenuiorum on account of the grade of society from which their membership was drawn and also called Collegia Funeraticia or burial societies, where, as was usually the case, the death benefit was the predominant feature, were not the only societies of ancient days which involved the life insurance feature. There appear to have also been societies among the soldiers with similar benefits on a somewhat larger scale but associated with other benefits more expensive. For example, one such society of which there is record paid about seventy dollars in case of death on service. The same

amount was also paid in case of promotion to a higher rank, transfer to another legion or retirement. The consideration for all of this seems to have been an admission fee of one hundred and five dollars, possibly payable in instalments.

From the nature of the case these associations were not organized on a scientific basis nor carried on in a systematic manner so that they sometimes failed just as institutions similarly organized on a larger scale have failed in modern times.

The principle of cooperation and mutual support, in the face of contingencies that are unforeseen or overpowering, which underlies all forms of insurance is very strongly illustrated also by the gilds which were so important a feature of the middle ages commencing with the Frith Gilds of early times, which had many analogies to the vigilance committees of our frontier days, being voluntary associations of neighbors for that mutual protection and support which the government of the day could not be depended upon to give, and developing through the merchant gilds, the craft gilds and the social and religious gilds. Even in the case of the latter, however, the relief given in case of death was not for death as such but was conditioned on resulting destitution and did not, therefore, exhibit the principle of life insurance as distinctly as did the Roman burial societies. They represented, in fact, insurance against destitution of self or family instead of insurance against any of the contingencies which might cause financial loss.

While, however, the Roman *Collegia* and the mediaeval gilds illustrate the fundamental social value of insurance and especially of life insurance, it is to a different organization that modern life insurance companies are directly linked. The loss sustained by the merchant and shipowner, the two being frequently if not usually in early days one person, must always have been a staggering one. It is only natural, therefore, that marine insurance should have been the first

form of insurance to be practiced on an extensive scale. In the early days it appears to have been entirely, as it still si largely, undertaken by individuals not organized into a partnership or company but each separately assuming the amount of risk set opposite their signatures subscribed to the policy of insurance. Hence the origin of the name "underwriters" for those assuming insurance risks.

Out of marine insurance there grew a practice of also insuring the life of the merchant or of the master of the ship for the voyage. These early life insurance policies were probably looked upon as merely marine insurance policies, the perils of the sea being those principally in mind in connection with them and they were underwritten by individuals in the same way as the regular marine policies.

The earliest life insurance policy of which we have a record was dated June 18th, 1583. We haye a record of this policy because it was the subject of a trial in the English courts. It was made at the Office of Insurance "within the Royal Exchange" in London and insured the life of one William Gybbons in favor of Richard Martin for a period of twelve months from its date at a premium rate of eight percent. The insured died May 29, 1584 and claim was duly made for the amount insured but the underwriters denied liability on the ground that the insured had survived for more than twelve lunar months of twenty-eight days each. The Commissioners of the City of London, the Clerk of the Office of Assurance and finally the Judges of the Court of Admiralty all ruled against the underwriters and they were required to pay.

In passing it may be as well to note that the wording of this policy shows that life insurance transacted in this manner had for some time been a recognized practice. It says (modernizing some of the language) "It is to be understood that this present writing is and shall be of as much force, strength and effect as the best and most surest policy or writing of assurance which hath been ever heretofore

used to be made upon the life of any person in Lombard Street, or now within the Royal Exchange in London" and further on refers to "the use and custom of the said street or Royal Exchange".

The next decisive step towards the modern system of life insurance was taken when, in 1699, there was founded in London the first mutual contribution life office. It was called "The Society of Assurance for Widows and Orphans". The plan was to have a society of 2000 members each of whom paid 5 shillings toward each death claim and an additional 5 shillings once for all on admission and one shilling quarterly. The amount paid in case of death was the 5 shillings from each member, making £500 in all if the number of 2000 were complete, less a deduction of 3 percent for expenses. The provision for running expenses was thus the admission fees, the quarterly dues of one shilling from each member and 3 percent of the claim contributions. Immediately upon the payment of a death claim the 5 shilling contribution for the next claim was collected and it was also collected from each new member in addition to the admission fee.

Two other similar institutions were established in 1700 and 1704 respectively but all three and many others which had been floated in the meantime were swept out of existence when the South Sea Bubble burst in 1720.

An institution was, however, established on slightly different lines in 1706 which was able to survive the disastrous year 1720. It was called the Amicable Society for a Perpetual Assurance. The contributions of the members were fixed at £6.4s per annum and an admission fee of 10s. The number of members was fixed at 2000 which number appears to have been attained in the second year. The amount paid as a death claim depended upon the number of deaths in the calendar year. The provision was that in the first year of the Society one sixth of the total contributions was to be divided among the death claims of the year, in the second year, on the assumption of a

complete membership of 2000, £4000 was to be divided, in the third year £6000, in the fourth year £8000 and thereafter £10,000 each year. As the full contribution from the complete roll of members, after deducting 4 shillings per annum from each which went to the registrar, amounted to £12,000 this left a liberal margin for accumulation of surplus so that even though it suffered some financial reverses in its early years it was able to survive. In 1807 its business was placed on a more scientific basis and in 1866 after 160 years of existence it was amalgamated with another company which is still in existence. In order to complete this historical review of prescientific life insurance it may be added that in 1721 two corporations, the London Assurance Corporation and the Royal Exchange Assurance Corporation, which had been already authorized to transact marine insurance, received the additional power to accept life insurance risks. They transacted the business on the old lines which had been followed by individual underwriters. The standard premium seems to have been 5 percent for all accepted ages.

We have thus traced the history of life insurance up to the time when the science of life contingencies as an application of mathematics was developed. It is an interesting fact, however, that although, up to the present time, life insurance has been the most important business to which that science has been applied it was not in connection with life insurance that it was first developed but rather in connection with life annuities. It must be remembered that although life annuities are affected by the same contingencies of life or death as is life insurance and while at the present time the great bulk of the business of granting life annuities is carried on by life insurance companies they serve a fundamentally different purpose and had a radically different origin.

With respect to life annuities our search for origins leads us back, as in the case of life insurance, to ancient Roman days but instead of associations of slaves we have

in this case the testamentary dispositions of the wealthy who directed that certain legatees should receive out of their estate a fixed income for life. Out of this practice a problem arose when a law was enacted that at least one quarter of any estate should be distributed among the legal heirs. For the purpose of determining if this law was complied with it was necessary to have some rule for finding the commuted value of these obligations to pay an income for life and a schedule of values graded according to quinquennial groups of ages of the life tenant was actually drawn up and used for that purpose. We have no record, however, of the basis upon which the figures in that table were arrived at nor of any investigations which would furnish appropriate data, so that as far as we can judge it was purely empirical or hypothetical.

There are probably instances in all subsequent ages of such life interests or annuities arising out of wills but it was from a different source that the great impetus to the business of life annuities come.

In early days no distinction was made between excessive and moderate rates of interest for loans. Possibly there were no instances of a moderate rate but in any event all interest or usury as it was called was condemned. This condemnation was at first a purely religious one but at various times laws were enacted by the civil authorities in various countries prohibiting usury. In some cases this prohibition was general but in others it applied only to the Christian subjects of the state so that non-Christians and aliens were permitted to practice it. The economic tendency which these church edicts and civil laws were endeavoring to combat was, however, so strong that even the clergy seem to have in many cases engaged in the practice. Capital, however, in the hands of men of ordinary prudence prefers a legal investment if it can be found. A legal way of satisfying economic needs was accordingly sought and found in the purchase and sale of rent charges and life interests. These were at first always charged

against specific parcels of real estate the owner of which, desiring to raise an immediate sum of money for some purpose, would agree to a definite annual charge against the income from the property either in perpetuity or for the lifetime of one or more nominees. Later these annuities were sometimes issued on the personal credit of the granter and in such cases were almost necessarily either for life or for a fixed term of years.

The sale of annuities was not confined to individual property owners. As early as the eighth century monasteries and other religious bodies sold life leases of land or rents for life. This means of raising money became popular with public bodies and from about the end of the thirteenth century it was largely monopolized by them, both town authorities and governments of states taking part in it. Many towns, in fact, forbade the sale of annuities by private persons and while in this step they may have been actuated by the motive of removing competition there were probably also actual complaints of abuses in the private sale.

Annuities were issued on two or more lives payable as long as any of the nominees survived. The income might be payable to the purchaser or his heirs the lives nominated not being financially interest in the contract or it might be payable to the nominees in a definite order of succession or in the third place might be equally divided among the survivors or their respective nominators. This latter plan with a large number of nominees develops into the form of annuity plan associated with the name of Tonti and therefore called tontines.

It is evident that if such an annuity is established on say ten thousand lives of sufficient amount to give each ten dollars a year at the start it will give very large incomes to each surviver near the end. This form of contract has for the individual purchaser a very considerable gambling element and for the seller is very nearly equivalent to a perpetuity.

This form of contract was quite popular in the seventeenth century in Europe and is of interest to us from the fact that a feature developed in connection with it which, I believe, helped to prepare the way for the scientific study of life contingencies. Taking a glance forward to the deferred dividend or tontine dividend system of profit distribution formerly so popular in this country and comparing it with this tontine annuity scheme it appears that with anything of the tontine nature estimates of results are almost inevitably associated. The promoters of tontine annuity schemes sometimes put out such estimates showing out of a given number of annuity purchasers the estimated number surviving after various intervals and the consequent income received by each.

Unfortunately, however, for historical completeness there is, so far as I know, no record of any of them explaining the basis of the estimates. They did, however, undoubtedly accustom people to the idea of the possibility of estimating how many out of a given group of people now formed would remain alive at any given date in the future and how many would die in each year.

Having thus outlined the steps by which the force of circumstances prepared the problem of life contingencies for the attention of mathematicians let us now turn our attention briefly to the other side of the screen and observe how mathematical science was being prepared to attack the problem. This leads us to a consideration of the foundation of the theory of probabilities and although any remarks of mine on this subject before this audience may possibly be likened to "carrying coals to Newcastle", I shall, for the sake of completeness, ask you to bear with me for a few minutes.

Although there are slight hints of the subject in the writings of Cardan, Kepler and Galileo the definite origin of the science is usually, and with justice, attributed to Pascal. With his name is associated that of Fermat who was consulted by Pascal and who worked out an independent

solution of the problem under discussion. It is interesting to note that at the date, namely 1654, when they discussed this problem and so laid the foundations of the theory, Pascal and Fermat were the most distinguished mathematicians of Europe, Wallis in England being practically their only rival. Descartes had died in 1650, Newton was only twelve years old, and Leibnitz eight. Another interesting fact is that both of these founders of the science might be classed as amateur mathematicians, by which I mean that mathematics was to each of them an avocation or hobby and not a source of income. Pascal had just the year before been called from his religious retreat at Port Royal to administer his father's estate and was at the time occupied with that business. He returned to the retreat in November of 1654. Fermat was and had been for twenty-three years councillor for the local parliament at Toulouse. He was at the time about fifty-three years of age and Pascal was thirty-one.

A third interesting fact is that the problem which brought the subject to Pascal's attention was itself of comparatively little importance and arose out of a game of chance. This may perhaps be considered natural because the subject matter of the science is chance and games of chance, at that time at least, received more attention than any other chance events.

The problem was proposed to Pascal by the Chevalier de Meré, who seems to have been addicted to games of chance, and was in substance as follows: Two players want each a given number of points in order to win. If they separate without playing out the game, how should the stakes be divided between them? It will be seen that this is equivalent to determining their respective chances of winning the game. This problem was solved on the assumption that the two players had equal chances of winning any particular point and that this was true regardless of who may have won any other point.

The methods followed by the two mathematicians in attacking the problem differed but their fundamental

assumptions were the same and the results consequently the same.

Pascal applied the method to some allied problems and intended to write a treatise on the subject but the accident which was the occasion of his return to the Port Royal retreat occurred shortly after and he does not appear to have ever carried out his intention. He died in 1662 at the age of 39. Todhunter in his *History of the Theory of Probability* says

"It might have been anticipated that a subject interesting in itself and discussed by the two most distinguished mathematicians of the time would have attracted rapid and general attention; but such does not appear to have been the case. The two great men themselves seem to have been indifferent to any extensive publication of their investigations; it was sufficient for each to gain the approbation of the other. Pascal finally withdrew from science and the world; Fermat devoted to mathematics only the leisure of a laborious life, and died in 1665".

The work of systematic presentation was taken up by Christian Huygens who wrote in 1657 a small treatise on the theory of probabilities founded on the correspondence of Pascal and Fermat. It was first published in Latin at the end of a collection of mathematical exercises by Frans van Schooten. This little treatise is notable because for a period of more than fifty years it formed the best account of the subject and it was during this fifty year period that the foundation of actuarial science was laid down. It is interesting to note that van Schooten's collection was one of the books read by Newton as an undergraduate at Cambridge. It would seem probable that Halley was also familiar with it or at least with the treatise by Huygens in view of the fact that a translation of it was included in an anonymous essay edited by Motte and printed in England in 1692. It was in this treatise of Huygens that the term *expectation* in the sense in which it is used in probabilities was first introduced.

Although, contrary to the usual opinion, the *expectation of life* is not a fundamental element of actuarial computations and is in fact merely a convenient basis of general comparison between different mortality tables the idea of *expectation* as developed in the theory of probability is fundamental. Its explicit introduction in this treatise is, therefore, worthy of note. The idea itself was, of course, implicit in the original problem proposed to Pascal. The quantities sought in that problem were the respective expectations of the players if play had been continued.

The calculus of probabilities, young and neglected as it was, had now reached a stage where it was ready for application to the problems of life contingencies and although the birth of actuarial science is usually referred to the first known calculations involving a combination of the theories of compound interest and of probabilities it may be well in passing to mention a development which immediately preceded that event. I refer to the publication in 1661 by Captain John Graunt, a London merchant, of a book entitled *Natural and Political Observations on the Bills of Mortality.* In this book, besides giving valuable statistics of deaths analyzed by cause and by age, an effort was made to draw conclusions regarding population and rates of mortality, including a table showing how many in each hundred die within the first six years of life and how many in each decade thereafter. This table was used twenty-one years later in connection with a tontine scheme as furnishing an outline mortality table. Considerations which must be taken into account if a mortality table anywhere near correct is to be obtained were apparently ignored and the table has no claim to accuracy as a representation of the rates of mortality. It will be interesting to compare the percentages of those entering each age interval who die in that interval according to this table with those according to the Carlisle Table which was the first based upon a comparison of deaths analyzed by age with the corresponding population. They are as follows:

Age Interval	Percentage of Death	
	Graunt	Carlisle
0–6	36.0	33.2
6–16	37.5	6.2
16–26	37.5	6.8
26–36	36.0	9.1
36–46	37.5	12.2
46–56	40.0	14.1
56–66	50.0	27.7
66–76	66.7	47.7
76–86	100.0	75.8

It will be noted that in Graunt's table these percentages are substantially the same for the five decennial intervals from age six to age fifty-six and correspond with an annual rate of about forty-six per thousand, a rate not reached in the Carlisle table until age sixty-eight.

While this book is clearly entitled to first place in the history of vital statistics it is difficult to define its relation to our subject because we cannot trace any definite connection between it and the preceding work of Pascal, Fermat and Huygens in the calculus of probabilities on the one hand nor between it and the subsequent actuarial developments on the other.

We now come to the definite starting point of actuarial science. It is a curious fact, however, that in one sense this starting point is not definite because the first piece of actuarial work on record has apparently no connection with the subsequent development of the science, which, in fact, followed continuously from an independent origin. This was partly due to special circumstances but the whole incident clearly demonstrates the value of scientific societies for the preservation and dissemination of the results of original research.

The importance of these incidents in connection with our subject will perhaps excuse some reference to the individuals concerned, as that will give a clearer idea of the circumstances. The author of the first actuarial report

was John deWitt, Grand Pensionary of Holland from 1653 to 1672. He was born at Dort in 1625, his father being a prominent leader of the party which was opposed to the extension of the executive power then vested in the Prince of Orange as elective stadtholder. The position of stadtholder of the United Provinces was in practice hereditary in the house of Orange but a formal election was necessary at each vacancy and when in 1650 the stadtholder, William II, died, leaving only a posthumous son born one week after his death, the post of stadtholder was left in abeyance and the chief executive authority devolved on the Grand Pensionary of the Province of Holland. Young John deWitt was elected to this position in 1653 at the age of twenty-eight and being re-elected at five year intervals he continued as the head of the government until 1672. Then when the young prince, William III of Orange, afterwards King William III of England, had attained his majority and the country was engaged in war against France and England a violent reaction not only drove John deWitt from power and installed William III as stadtholder but resulted in the murder of John deWitt and his brother Cornelius by a mob on August 20, 1672.

Before his election as Grand Pensionary he had already in 1650, displayed his interest in mathematics by writing a treatise on curved lines and it was natural, therefore, that he should take a personal interest in the questions of a mathematical nature which came up during his administration.

Reference has already been made to the custom of municipalities and other governments raising money by the sale of life annuities, the purchaser being allowed to nominate the person on whose life the annuity was to depend. The ordinary terms appear to have been that a life annuity was considered as worth one half of the value of a perpetuity of the same annual amount so that, for example, on a four percent basis, a perpetuity being worth

twenty-five years' purchase, a life annuity would be worth twelve and a half years' purchase. In the case, however, of the annuities sold by the government of Holland this seems in deWitt's time to have already been modified so as to require fourteen years' purchase instead of twelve and a half.

The object of deWitt's report was to show that a life annuity at fourteen years' purchase yielded a better return to the purchaser than a redeemable annuity or perpetuity at twenty-five years' purchase. This he does by calculating the value at age three, on a four percent interest basis, of a life annuity of 20,000,000 stuyers per annum payable half yearly. He assumes that the same number die in each half year for fifty years from age three to age fifty-three, then two-thirds as many each half year for the next ten years, one half as many for the next ten years and one third as many for seven years and that all are dead at age eighty. His method of calculation is to multiply together the number assumed to die in each half year and the present value of the annuity they will each have drawn and to divide the sum of these products by the total number. It will be seen that this is a direct calculation on the principles of the theory of probability of the expectation of the purchaser.

The report itself does not attempt to justify the assumptions made regarding the distribution of deaths but in subsequent correspondence with John Hudde, Burgomaster of Amsterdam, and also of some note as a mathematician, he indicates that they were based in some way on observations. He states that from further observations he believes that the uniform decrements should extend up to age seventy-five.

The value which he had arrived at for a life annuity at age three was just over sixteen years' purchase and in a supplementary report he indicates a practical check which he had applied. He says that he had extracted from the records the data regarding several thousand purchasers of annuities and had calculated the average value of the

annuities drawn in each of considerably more than one hundred classes of about one hundred lives each and that in no case had he found the average value less than sixteen years' purchase but they were frequently much higher, even up to eighteen years' purchase. Only a few copies of this report were printed and, possibly due in part to the troubles of the following year, when deWitt was driven from power, all trace of the report was lost although the fact was known that it had been made and it was not recovered until the middle of the last century, when, through the researches of Frederick Hendriks, a copy was found in the archives of the Netherlands.

The next to venture in this field, while not so prominent politically as de Witt, was much more prominent scientifically. I refer to the famous astronomer Edmund Halley. It is unnecessary to recite to this audience the fact that he was the first to calculate the periodic time of a comet nor the great service to mathematical and physical science that he rendered by inducing Newton to write his *Principia*. The whole cost of its publication was borne by Halley who also corrected the proofs.

If the results which flowed from Halley's work illustrated as we have already pointed out the value of scientific societies for the preservation and publication of original work his actually undertaking it illustrated their value in actually stimulating such work. He was at the time actively interested in the work of the Royal Society and in order to insure the continuation of the publication of its TRANSACTIONS he undertook at a meeting of the council in December, 1692, to furnish five sheets in twenty himself. It was probably under the spur of this promise that he took up the analysis of the data regarding births and deaths in the city of Breslau which had just recently been contributed to the Society and the result was a paper entitled *An estimate of the degree of mortality of mankind, drawn from curious tables of the births and funerals at the city of Breslau; with an attempt to ascertain the price of*

annuities upon lives which was presented early in 1693. It is notable as containing the first known example of a complete mortality table. This table was constructed from the deaths during the five years 1687 to 1691 inclusive on the assumption that the population was stationary and that consequently the number attaining any age would equal the number of deaths above that age. It was not quite in the usual modern form because the number shown opposite each age was not the number attaining that precise age but the number living at any given moment in that year of age, for example, opposite age 25 we find the population at age 25 next birthday. These figures can of course be used in the calculation of annuity values just like the usual number living (l_x) column of a mortality table provided we remember that the value so arrived at corresponds to age next birthday and not to exact age.

Among the uses for this table suggested by the author are the determination of the premiums for what we now call one year term insurance and the valuation of annuities on one life, on two lives and on three lives. His formula for the value of a life annuity is the one still used in explaining the elementary theory of the subject and represents the sum for positive integral values of n of the products of the present value of a unit due n years hence and the probability that the life or lives on whose survival the payment was conditioned would survive the n years. For an annuity on one life this was mathematically equivalent to the formula used by de Witt but was simpler in practical operation and more easily applied to two or more lives. In a subsequent communication he points out the great advantage of using logarithms for these calculations. The more practical methods proposed by Halley and this later suggestion regarding logarithms shows the effect of his experience with arithmetical calculations in connection with his work in astronomy.

The work of these two independent pioneers brings out the fact that the work of actuarial science like that of

every other branch of applied mathematics falls into two main divisions. The first is the ascertainment of those facts of nature which can be used as premises for the mathematical calculations, in this case primarily the facts regarding human mortality, although the rate of interest appropriate to the particular problem in hand and the rate of expense necessary to be provided for are also factors. The second is the development of the mathematical processes best suited to the problems arising and in this division economy of labor is second in importance only to substantial accuracy of result. It may be of interest to outline the main problems which come up for consideration in each of these divisions, merely prefacing the whole with the remark that there is a very considerable mathematical element in the first division as well as in the second.

It might appear that the straightforward way of constructing a mortality table from actual experience was to take note of a large number of babies born about the same time and have them all observed until the last one died, noting the number dying in each year of age. The mere statement of the process, however, shows that it is impracticable. Some other process is therefore necessary and it was perhaps fortunate for the business of life insurance that the methods first followed erred in the direction of too high rates of mortality so that the premiums charged were higher than proved ultimately necessary. Graunt's figures were apparently a simple analysis by age of the deaths reported in London during a few years. The basis of deWitt's calculations was stated as a pure hypothesis without any attempt at justification in detail. Halley apparently recognized the disturbing effect of migrations and of a rapidly changing population. Owing, however, to the fact that the recorded births in Breslau only slightly exceeded the recorded deaths he assumed that the population might be considered stationary. He, therefore, graduated by inspection the average number dying per annum at the various ages and assumed that

this gave the decrements of the table, the number living being given by summing from the oldest age downwards. The next important landmark is from one point of view a backward step, appearing to substitute hypothesis for observation, but from another point of view it might be considered as the first attempt to graduate a mortality table by a mathematical law. I refer to deMoivre's hypothesis that the number dying was the same at all ages with eighty-six as the limiting age. I refer to it as a graduation because the smooth series of the hypothesis agreed in a general way, except in infancy and extreme old age, with Halley's and later tables constructed on the same principle that were current at the time he propounded it. This brings out the point that the first division of subject consists of two parts, the ascertaining of the crude facts observed and the adjustment of the results. In fact, Halley in constructing his table had not used the schedule of deaths exactly as reported but had substituted a smoother series.

With regard to the ascertaining of the crude facts the greatest advance was made with the construction of the Carlisle table by Milne because he was able to put into effect the principle that in the construction of a mortality table the fundamental functions are the rates of mortality at the successive ages and not the deaths at the various ages. These rates of mortality he was able to calculate because he had as data not only the deaths in the parishes of St. Mary and St. Cuthbert, Carlisle, for the nine years 1779 to 1787 classified by groups of ages but also the population January 1st, 1780, classified in substantially the same way and an estimate at least of the increase in population up to December 31st, 1787. I speak of it as an estimate at least because the figures used indicate an increase of exactly 1000. It is interesting to compare these various tables and hypotheses including the Northampton table constructed by Dr. Price from the deaths in the parish of All Saints, Northampton, during the forty-six

years from 1735 to 1780 inclusive. I have chosen for the purpose of comparison the probability of a child aged six years dying within thirty years and the similar probability at age forty-six. The comparison is as follows:

	Percentage dying within 30 years	
	Age 6	*Age* 46
Graunt	75.0	90.0
deWitt	49.2	92.7
Halley	32.0	80.9
DeMoivre	37.5	75.0
Northampton	35.1	73.8
Carlisle	20.5	67.5

The marked excess of the early estimates of mortality at young ages is very noticeable.

The problems involved in the tabulation of the unadjusted rates of mortality derived from returns relating to the general population differ in many particulars from those arising in connection with insurance experience. On the former the returns since the day of systematic periodic censuses and fairly accurate death registration furnish ample data and the special problems arose at first from the fact that the age classification was only made for groups of ages and the principles of finite differences had to be applied to obtain the rates for individual ages. In later years when returns were made for individual ages we are confronted by the fact that people have a bias in favor of round figures of various kinds in stating their ages so that a careful regrouping is necessary to minimize the error from this source. Also experience indicates a deficiency in the enumerated population in the first two years of age and the question of an approximate correction of the deficiency is an interesting one. In insurance experience the ages are more accurately known and the primary problem is the practical treatment of those withdrawing from the field of observation otherwise than by death. As the requirements of the business, however,

progress we find that a simple classification by attained age is not sufficient. We must investigate special classes of risk on account of sex, residence, occupation, physical impairment or other special features and in doing so must take account of the fact that the rate of mortality depends also on the time elapsed since the life was accepted for insurance. In other words for insurance purposes a man is not simply a man of a given age but we know that other ascertainable features affect the probabilities of death.

The graduation or smoothing of the crude rates of mortality is justified by the fact that according to the theory of probability the actually observed ratios of death are not likely to agree exactly with the true probability and indeed cannot unless the product of the probability and the number at risk happens to be an integer. Also a fluctuation in one direction at a given age is as likely as not to be followed by a fluctuation of opposite sign at the next age. An actually observed irregular series is therefore not inconsistent with the natural assumption that the series of true probabilities is a smooth one. Various methods have been used to graduate mortality tables. Halley's table and the Northampton table were graduated roughly by simple inspection. Milne used a graphic method to distribute the population and deaths into individual years in such a way as to preserve the totals of each group of ages unchanged. We have already mentioned deMoivre's hypothesis as possibly entitled to be considered a sort of graduation by mathematical law but the law most extensively applied to the graduation of mortality tables was Makeham's modification of Gompertz' law. Gompertz originally suggested that the force of mortality or instantaneous death rate increased in geometrical progression with the age. It was, however, found impossible to use a single rate of progression throughout even adult life so that it was necessary to break the table into two parts using one rate for one part and another for the other. To overcome this difficulty Makeham proposed breaking

the force of mortality into two parts, one of which remained constant and the other increased in geometrical progression. With this modification it was found possible to reproduce with a fair degree of accuracy the mortality indicated by a number of different tables from an age in the twenties to the limit of life. This law possesses advantages in connection with joint life calculations so that it is desirable to use it if possible but recent experiences have been rather intractable. In the case of the experience on American insured lives recently published by the Actuarial Society it was necessary to introduce an additional term into the expression for the force of mortality in order to fit the table from age 30 up. This term was negative in simple proportion to the age.

Other formulas have been used, some of them very elaborate, but no particular one has been used generally and none of them possess the practical advantage of Makeham's.

A third general method of graduation is based on the principle of moving average. As first applied it consisted in substituting for each term of the series the average of five consecutive terms of which it was the central and then repeating the process a second and third time. This, however, would introduce an error into a regular second difference series so a correction is introduced to eliminate this error so that a regular third difference series would be reproduced. Many variations have been suggested and the investigation of the smoothing power and residual error of the various formulas is an interesting problem. A fourth plan which has been used is to apply the principles of finite differences to determine adjusted values for the exposed and deaths at quinquennial ages and then having calculated from these the rates of mortality at those ages to fill in the remainder by interpolation.

A new method of graduation has recently been put forward by E. T. Whittaker of Edinburgh which consists in determining a graduated series of values such that the

sum of the squares of the third differences of those values plus the sum of the squares of the differences between the corresponding graduated and ungraduated terms multiplied by a selected constant shall be a minimum. This requirement gives a linear difference equation of the sixth order with end conditions sufficient to determine uniquely a particular solution which is the graduated series required. It is evident that the graduating power depends on the constant selected. The smaller the value of the constant the smoother the resulting series and the greater the sum of the squares of the departures of the graduated from the ungraduated values. For all values of the constant the algebraic sum of the departures and of those departures multiplied by the respective ages or by the squares of the ages will vanish.

Taking up now the other division of our science, namely, the development of the mathematical processes suited to its problems, it will be recalled that the methods of Halley brought out the element so frequent in actuarial calculations, of the determination of the value of an expectation. Where the benefit as in the case of a life annuity payable annually consists of payments to be made on the anniversaries of the date of calculation provided specified contingent events have happened this is theoretically simple. The only difficulties arise from the complicated probabilities sometimes met with. The case of life insurance is ordinarily brought under this heading by assuming that the sum insured is payable at the end of the year in which death occurs. As the possible future life time extends over a considerable number of years the preparation of a table of values if calculated independently is rather laborious. DeMoivre accordingly showed how the value of a life annuity at any given age may be derived by a short calculation from that at the next older age and a similar principle may be applied to other values. His formula is still of theoretical value and of occasional practical use but for general purposes has been superseded by the device of commutation columns, the fundamental idea of which is the discounting

back not to the present time but to the date of birth of the total payments called for if a contract of the kind being valued were in force on every life represented by the number living according to the mortality table used. The functions, therefore, which enter into valuation at one age enter largely into the value at another age and certain functions are found of very wide availability thus yielding a great economy of labor.

The principles of finite differences or of the infinitesimal calculus come into play in cases of two kinds. In the first case periodical or other payments may be payable otherwise than annually on the anniversary and it may be desired to derive formulas as accurate as possible for the present values not necessarily for actual use but in order to know the limits of error in the simpler formulas actually used. For such cases formulas have been derived for the correction to pass from the annual case to the fractional. Again the probabilities involved in a particular problem may be so complicated that their calculation for every year of possible future duration of the contract would be too laborious. In such cases they are calculated only at intervals and the principles of finite differences applied to obtain approximate values on the assumption that the probabilities vary regularly in the intervals.

In this connection it is interesting to note that the development of the theory of osculatory interpolation has come almost entirely from the needs of life insurance calculations. The curves used for interpolating in the successive intervals are so adjusted as to pass smoothly into one another without using too high an order of differences. Such a method was more necessary in life insurance than in astronomical work because the basic values which must be used upon which to interpolate are not quite as regular as the calculated values occurring in astronomy.

I have thus outlined briefly and, let us hope, not too tiresomely the nature of the mathematical processes required for actuarial calculations. Let us review a few of the

results that have followed the development of the business.

Life insurance still serves in increasing measure the special purposes which called it into existence and as it was the demand for what we would now call business insurance which had sufficient financial importance to call for the establishment of life insurance corporations so it is appropriate that to-day the same demand should call for the largest policies and furnish the most interesting news items. Large amounts of insurance are called for by wealthy people to cover inheritance taxes or to provide ready money to meet the obligations likely to mature at their death. Partnerships ask for it to provide ready money to settle with the estate of a dead partner without sacrificing the business. Businesses which are largely built about the skill, technical knowledge or executive capacity of some individual demand insurance to compensate in a measure for their loss. The purpose, however, of life insurance which is of the greatest social importance is to continue in part and for a time at least to the dependents of the insured the income and support which would otherwise cease at death. To this has recently been added the furnishing of a similar support to the insured himself and his family should he suffer from total and permanent disablement which is, financially at least, worse than death. It has also developed the service which it offers freely to the beneficiaries of life insurance policies of holding the proceeds, investing them for the beneficiaries and paying principal and interest in such convenient manner as may have been arranged thus relieving a beneficiary, probably inexpert and certainly unprepared, from the trouble and risk of seeking investments.

An additional important social service performed by life insurance arises from the necessity of finding investment for the reserve funds which the companies must accumulate and upon which they must earn interest. I might dwell at length on the way in which life insurance funds have aided in the development of our farms and railways and

in the building of our cities but this feature of its service has been put so much better than I could do it, by the late Mr. Peter McArthur, that I conclude by quoting some extracts from a pamphlet which he published a few years ago. He says:—

"In this time of discontent it should be made clear that no matter what accusations are brought against wealth, insurance money is unselfish money. This has not been clear in the past. Possibly it has not always been true. But insurance funds have now reached such proportions that their real nature is becoming evident. The inherent laws that govern insurance money are being released and their power is becoming apparent. In spite of what any man or group of men may do these great accumulations of the savings of the people are bound by their very nature to be on the side of the people and to work for them. As they go on increasing they will do more and more to safeguard the rights of humanity. In short, I have no hesitation in saying that insurance is now the most steadying force in our civilization.

"It is estimated that in the United States the established fortunes that are devoted to investment rather than to active enterprise amount to something over $15,000,000,000. The insurance funds already equal fully one-half of this amount and at their present ratio of increase will probably exceed it in a few years. In Canada the conditions are much the same. While no man can predict what will be the entire result of this competition between the great fortunes and the insurance companies for possession of the most desirable securities, the power of individual wealth will certainly be curtailed and the safety of the people will be increased. This is the first time in history that anything of this kind has occurred. In the gamble of life the percentage was always in favor of those who had reserves of wealth. Now, however, life insurance is putting the percentage in favor of the ordinary people."

EQUITABLE LIFE ASSURANCE SOCIETY

SIMPLIFICATIONS RELATING TO A PROOF OF SYLOW'S THEOREM

BY G. A. MILLER

Sylow's theorem is usually developed very early in a course relating to the theory of groups of finite order. Hence simplifications in its proof are the more desirable, especially when they enable us to avoid the use of a number theory formula and therefore to confine the proof more closely to group theoretic considerations. Since the simplifications which will be developed in what follows involve a property of double cosets which is not directly used in the proof to which they relate* we shall first exhibit some of the fundamental properties of double co-sets, assuming as known the fact that if H_1 and H_2 are any two subgroups of a group G then all the operators of G may be uniquely represented in the following form. $G = H_1 s_1 H_2 + \cdots + H_1 s_\lambda H_2$ if we assume that in each double co-set only the distinct operators are considered.

The number of distinct operators in each double co-set is evidently divisible by h_1 and by h_2, h_1 and h_2 being the orders of H_1 and H_2 respectively. A necessary and sufficient condition that each of the given double co-sets contains the same number of distinct operators is that each of the conjugates of H_1 under G has the same number of operators in common with H_2. This is equivalent to saying that each of these conjugates of H_2 has the same number of operators in common with H_1. When this number of common operators is k then the number of distinct operators in each of these co-sets is $h_1 h_2/k$, and vice versa. In particular, each of these double co-sets must involve the same number of distinct operators whenever at least one of the two subgroups H_1, H_2 is invariant under G, and each of the double co-sets with respect

* Miller, Blichfeldt, and Dickson, *Finite Groups*, 1916, p. 27.

to H_1 and H_2 involves exactly $h_1 h_2$ distinct operators whenever h_1 and h_2 are relatively prime. When H_1 and H_2 belong to the same set of conjugates under G then the number of these double co-sets which involve exactly h_1 distinct operators is evidently equal to the index of H_1 under the largest subgroup of G which transforms H_1 into itself, and all the other double co-sets involve a multiple of h_1 distinct operators. All of these multiples must be sub-multiples of h_1^2. When H_1 and H_2 are identical it is obvious that they may be regarded as conjugate, and hence the theorem which has just been stated applies directly to the double co-sets with respect to a single sub-group.

Suppose now that $h_1 = p^\alpha$, where p is a prime number, and assume that the index of H_1 under the largest subgroup of G which transforms H_1 into itself is k, k being prime to p. If the operators of G are represented as double co-sets with respect to H_1, i. e., if H_2 is assumed to be identical with H_1, it results that the order g of G can be represented as follows: $g = kp^\alpha + lp^{\alpha+1}$. Hence p^α must be the highest power of p which divides g. If c represents the number of conjugates of H under G it results that $kp^\alpha = g/c$, and hence $(c-1)g = lp^{\alpha+1}$. That is, $c \equiv 1 \bmod p$. It has therefore been proved by means of double co-sets that if H_1 is of index prime to p under the largest subgroup of G which transforms H_1 into itself then H_1 is a Sylow subgroup of G and the number of its conjugates under G is congruent to 1 mod p.

From the preceding paragraph it results that a necessary and sufficient condition that a subgroup H of order p^α is a Sylow subgroup of G is that the index of H under the group K formed by all the operators of G which transform H into itself is prime to p, for if this index were not prime to p the quotient group K/H would have an order which is divisible by p. Since the order of this quotient group is less than g it may be assumed that this group contains a subgroup whose order is a power of p and hence G contains a subgroup of order $p^{\alpha+1}$. This

suggests that a proof of Sylow's theorem as regards the symmetric group S of degree p^m can be based upon a proof of the fact that S involves a subgroup P of order p^α which is of index prime to p under the group formed by all the substitutions of S which transform P into itself. This fact can easily be proved by mathematical induction.

When $m = 1$ the theorem is evident. If we assume that the theorem is true for the symmetric group of degree p^{m-1} it is not difficult to prove it true for S. In fact, S obviously contains a substitution s of order p and of degree p^m. The p^{m-1} cycles of this substitution are evidently transformed according to the symmetric group of degree p^{m-1} by the largest subgroup of S which transforms s into itself. The substitutions of S which transform the cycles of s according to a Sylow subgroup in the symmetric group of degree p^{m-1} generate a group of order p^α which contains no invariant substitution besides those generated by s. Hence all the substitutions of S which transform this group of order p^α into itself also transform s into itself, and therefore this group of order p^α is of index prime to p under the group formed by all the substitutions of S which transform it into itself. This group must therefore be a Sylow subgroup of S. From the fact that S contains a Sylow subgroup of order p^α it is very easy to deduce that every group whose order is divisible by p contains at least one Sylow subgroup whose order is a power of p.

The considerations which precede prove incidentally that the Sylow subgroups of order p^α contained in S are of index $(p-1)^m$ under S, since the index of these Sylow subgroups is obviously $p-1$ times the index of the corresponding Sylow subgroups under the symmetric group of degree p^{m-1}. Hence we not only know that this index is prime to p but we also know its exact value. The said considerations also prove that the number of the Sylow subgroups of order p^α contained in S which have in common a given invariant subgroup of order p is equal to the number of the Sylow subgroups of order a power of p in the sym-

metric group of degree p^{m-1}. Moreover, the number of the Sylow groups of order p^α contained in S is this number times the number of the possible subgroups of order p and of degree p^m contained in S.

From the method of proof here employed it also results directly that if the largest possible subgroup common to two Sylow subgroups of order p^α in any group G is of order p^β then the number of these Sylow subgroups is $\equiv 1 \bmod p^{\alpha-\beta}$. For instance, it is well known that no two subgroups of order 4 contained in the simple group of order 60 have any operator in common besides the identity. Hence the number of these subgroups must be $\equiv 1 \bmod 4$. As a matter of fact it is 5. It should be added that the simplifications suggested above do not apply to the well and favorably known proof of Sylow's theorem due to G. Frobenius. They apply to the older proof based on some properties of the symmetric group, especially to the form in which this proof is developed in the work to which reference was made in the first paragraph.

In closing we wish to refer to a variation in the proof of another fundamental theorem in the theory of groups of finite order since this variation may possibly simplify the proof for some readers. This variation relates to a proof that every subgroup of index 2 of any group G is invariant under G. To prove this theorem it may first be noted that if s represents any operator of such a subgroup H while t represents any operator of G which is not contained in H then st cannot be in H as otherwise the equation $xy = z$ would have more than one solution in G when two of its symbols are replaced by operators of G. For the same reason the product of two operators of G, neither of which is in H, must be in H. As $t^{-1}st$ is such a product, since neither st nor t^{-1} is in H, $t^{-1}st$ is such a product and must be in H. That is, H must be an invariant subgroup of G as a result of the fact that it is composed of half the operators of G.

THE UNIVERSITY OF ILLINOIS

REDUCTION OF EULER'S EQUATIONS TO A CANONICAL FORM*

BY J. H. TAYLOR†

1. *Introduction.* As a by-product of the preparation of an earlier paper by the author, *A generalization of Levi-Civita's parallelism and the Frenet formulas,* Dissertation, University of Chicago, 1924,‡ two useful methods of solving for the second derivatives in Euler's equations associated with the problem of minimizing an integral were discovered. In this paper these two methods are presented in detail. Whereas it appears at first that the assumptions required to effect the solution in the two instances are quite different, it is here shown that in each of the two cases the assumptions which are made may be replaced by the supposition that the F_1 function of the calculus of variations does not vanish.

2. *The Euler Equations.* Suppose that

$$x^\alpha = x^\alpha(u), \quad u_1 \leqq u \leqq u_2, \qquad (\alpha = 1, \cdots, n),$$

are the equations of one of a class of curves joining two fixed points in an n-space, and let us consider the problem of selecting that curve of the class which gives to the integral

$$I = \int_{u_1}^{u_2} F(x, x')du$$

its minimum value. Here x and $x' = dx/du$ stand for the sets $x^1, \cdots, x^n$ and $x'^1, \cdots, x'^n$ respectively. It will be assumed that F satisfies the homogeneity condition

$$F(x, \varkappa x') = \varkappa F(x, x'), \qquad \varkappa > 0. \tag{1}$$

* Presented to the Society, October 25, 1924.

† National Research Fellow in Mathematics.

‡ TRANSACTIONS OF THIS SOCIETY, vol. 27 (1925).

As a consequence of this it follows that

$$F_\alpha x'^\alpha = F, \quad F_{\alpha\beta} x'^\alpha = 0, \qquad (\beta = 1, \cdots, n), \tag{2}$$

where the notation is $F_\alpha = \partial F/\partial x'^\alpha$, $F_{\alpha\beta} = \partial^2 F/\partial x'^\alpha \partial x'^\beta$, and where we employ the convention that each index letter which appears twice in the same term is understood to indicate a summation with respect to that letter from 1 to n as is customary in tensor analysis. From the second of equations (2) it follows that unless $x' \equiv 0$, the determinant $|F_{\alpha\beta}| = 0$.

The Euler differential equations defining the extremals for the integral I are

$$\frac{d}{du}F_\beta - \frac{\partial F}{\partial x^\beta} = F_{\beta\alpha}x''^\alpha + \frac{\partial F_\beta}{\partial x^\alpha}x'^\alpha - \frac{\partial F}{\partial x^\beta} = 0, \qquad (\beta = 1, \cdots, n). \tag{3}$$

These equations are not independent but satisfy the relation

$$\left(F_{\beta\alpha}x''^\alpha + \frac{\partial F_\beta}{\partial x^\alpha}x'^\alpha - \frac{\partial F}{\partial x^\beta}\right)x'^\beta = 0, \tag{4}$$

which is easily verified by differentiating the identity $F_\beta x'^\beta = F$ with respect to u and applying (2). Let us suppose now that the parameter u has been so chosen that along our solutions of equations (3) we have $F(x, x') = 1$. Clearly such a selection is always possible if $F \neq 0$ along the solution, as we shall suppose, with the aid of the homogeneity condition (1). Since the determinant of coefficients $|F_{\alpha\beta}|$ vanishes we cannot solve (3) for the second derivatives by the usual process. Consider then the solution of the following system of equations

$$\begin{cases} F_{\beta\alpha}x''^\alpha + F_\beta w + \dfrac{\partial F_\beta}{\partial x^\alpha}x'^\alpha - \dfrac{\partial F}{\partial x^\beta} = 0 \\ F_\alpha x''^\alpha + \dfrac{\partial F}{\partial x^\alpha}x'^\alpha = 0. \end{cases} \tag{5}$$

It will be noticed that these differ from (3) in the introduction of an auxiliary variable w, and the adjoining of

an equation obtained by differentiating $F = 1$. We shall assume that the determinant of coefficients

$$(6) \qquad \begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & F_1 \\ F_{21} & F_{22} & \cdots & F_{2n} & F_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} & F_n \\ F_1 & F_2 & \cdots & F_n & 0 \end{vmatrix}$$

is not zero, and hence that the equations (5) may be solved for x''^{α}, w as functions of (x, x'). From (4) it follows at once that $w \equiv 0$, and therefore that the solutions x''^{α} of (5) are also solutions of (3).*

3. *The F_1 Function of the Calculus of Variations.* If we suppose the rank of $|F_{\alpha\beta}|$ to be $n-1$, we have from the second of (2)

$$\frac{x'^1}{F^{\beta 1}} = \frac{x'^2}{F^{\beta 2}} = \cdots = \frac{x'^n}{F^{\beta n}}, \quad (\beta = 1, \cdots, n),$$

where $F^{\beta\alpha}$ denotes the cofactor of $F_{\beta\alpha}$ in $|F_{\mu\nu}|$. On multiplying these ratios by x'^{β} and making use of the symmetry

$$F^{\alpha\beta} = F^{\beta\alpha}$$

it is seen that we may define a function F_1^* by the equations

$$\frac{x'^{\beta}x'^{\alpha}}{F^{\beta\alpha}} = \frac{1}{F_1^*}, \quad (\alpha, \beta \text{ not summed})$$

or

$$(7) \qquad x'^{\alpha}x'^{\beta}F_1^* = F^{\alpha\beta}.$$

The usual notation for this function is F_1, but F_1^* is used here to avoid confusion with $F_1 = \partial F/\partial x'^1$. From (7) it follows that

$$F_{\alpha}F_{\beta}x'^{\alpha}x'^{\beta}F_1^* = F_{\alpha}F_{\beta}F^{\alpha\beta},$$

* This method of solution was suggested by an analogous scheme used by Mason and Bliss, *The properties of curves in space which minimize a definite integral*, TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), p. 443.

and, by (2), this becomes

$$(F)^2 F_1^* = F_\alpha F_\beta F^{\alpha\beta}.$$

Hence

$$(8) \qquad F_1^* = \frac{-1}{(F)^2} \begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & F_1 \\ F_{21} & F_{22} & \cdots & F_{2n} & F_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} & F_n \\ F_1 & F_2 & \cdots & F_n & 0 \end{vmatrix}.$$

That is, the determinant (6) differs from the F_1^* function by a non-zero factor.

4. *A Second Method of Solution.* Let a function $f(x, x')$ be introduced by

$$(9) \qquad f(x, x') = \frac{1}{2} F^2(x, x').$$

The homogeneity condition is now expressed by

$$(10) \qquad f(x, \varkappa x') = \varkappa^2 f(x, x'), \qquad \varkappa > 0.$$

From (9) and (10) the following relations are readily deduced, the subscripts denoting as before the partial derivatives with respect to the corresponding x' variables:

$$(11) \quad \begin{cases} f_\alpha x'^\alpha = 2f, & f_{\alpha\beta} x'^\alpha = f_\beta, \\ \dfrac{\partial f_\alpha}{\partial x^\beta} x'^\alpha = 2\dfrac{\partial f}{\partial x^\beta}, & \dfrac{\partial f_{\alpha\beta}}{\partial x^\gamma} x'^\alpha = \dfrac{\partial f_\beta}{\partial x^\gamma}, \\ f_\alpha = F F_\alpha, & F_\alpha = \dfrac{f_\alpha}{F}, \\ f_{\alpha\beta} = F_\alpha F_\beta + F F_{\alpha\beta}, & F_{\alpha\beta} = \dfrac{f_{\alpha\beta}}{F} - \dfrac{f_\alpha f_\beta}{F^3}, \\ \dfrac{\partial f_\alpha}{\partial x^\beta} = F_\alpha \dfrac{\partial F}{\partial x^\beta} + F \dfrac{\partial F_\alpha}{\partial x^\beta}, & \dfrac{\partial F_\alpha}{\partial x^\beta} = \dfrac{\dfrac{\partial f_\alpha}{\partial x^\beta}}{F} - \dfrac{f_\alpha}{F^3} \dfrac{\partial f}{\partial x_\beta}. \end{cases}$$

The Euler equations are

$$F_{\beta\alpha} x''^\alpha + \frac{\partial F_\beta}{\partial x^\alpha} x'^\alpha - \frac{\partial F}{\partial x^\beta} = 0,$$

or

$$\frac{f_{\beta\alpha}x''^{\alpha}+\dfrac{\partial f_\beta}{\partial x^\alpha}x'^{\alpha}-\dfrac{\partial f}{\partial x^\beta}}{F}=\frac{f_\beta}{F^3}\left(f_{\alpha}x''^{\alpha}+\frac{\partial f}{\partial x^\alpha}x'^{\alpha}\right).$$

If again we suppose the independent variable to be so selected that $F=1$, the right member reduces to zero and the equation becomes

$$f_{\beta\alpha}x''^{\alpha}+\frac{\partial f_\beta}{\partial x^\alpha}x'^{\alpha}-\frac{\partial f}{\partial x^\beta}=0.$$

By (11), this may be written in the form

$$f_{i\alpha}x''^{\alpha}+\left(\frac{\partial f_{i\beta}}{\partial x^\alpha}-\frac{1}{2}\frac{\partial f_{\alpha\beta}}{\partial x^i}\right)x'^{\alpha}x'^{\beta}=0. \tag{12}$$

The question of the solution of these equations for the second derivatives fórces our attention upon the determinant $|f_{\alpha\beta}|$.

The determinant in question is

$$\begin{vmatrix} F_1F_1+FF_{11}, & F_1F_2+FF_{12}, & \cdots, & F_1F_n+FF_{1n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_nF_1+FF_{n1}, & F_nF_2+FF_{n2}, & \cdots, & F_nF_n+FF_{nn} \end{vmatrix}$$

Since

$$|F_{\alpha\beta}|=0,$$

the determinant reduces to

$$\begin{vmatrix} F_1F_1, & FF_{12}, & \cdots, & FF_{1n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_nF_1, & FF_{n2}, & \cdots, & FF_{nn} \end{vmatrix}+\begin{vmatrix} FF_{11}, & F_1F_2, & FF_{13}, & \cdots, & FF_{1n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ FF_{n1}, & F_nF_2, & FF_{n3}, & \cdots, & FF_{nn} \end{vmatrix}$$

$$+\cdots+\begin{vmatrix} FF_{11}, & \cdots, & FF_{1n-1}, & F_1F_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ FF_{n1}, & \cdots, & FF_{nn-1}, & F_nF_n \end{vmatrix}$$

$$=F^{n-1}[F_1F_\alpha F^{\alpha 1}+F_2F_\alpha F^{\alpha 2}+\cdots+F_nF_\alpha F^{\alpha n}]$$

$$=F^{n-1}F_\alpha F_\beta F^{\alpha\beta}.$$

Hence

$$|f_{\alpha\beta}| = -F^{n-1}\begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & F_1 \\ F_{21} & F_{22} & \cdots & F_{2n} & F_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} & F_n \\ F_1 & F_2 & \cdots & F_n & 0 \end{vmatrix},$$

which is the same as the F_1^* function except for a non-zero factor. If now we assume as before that the F_1^* function is different from zero, and therefore $|f_{\alpha\beta}| \neq 0$, the solutions of (12) may be obtained in the form

$$x''^{\lambda} = -\Gamma^{\lambda}_{\alpha\beta} x'^{\alpha} x'^{\beta}, \tag{13}$$

where $\Gamma^{\lambda}_{\alpha\beta}(x, x')$ is a function formed from $f_{\alpha\beta}(x, x')$ in precisely the same manner as the Christoffel symbol of the second kind $\{\alpha\beta, \lambda\}$ is formed from the quadratic form representing the square of the length element of a Riemann geometry. The explicit formulas for these coefficients are $\Gamma^{\lambda}_{\alpha\beta}(x, x') = f^{\lambda\mu}[\alpha\beta, \mu]$, where $f^{\lambda\mu}$ is the element of the reciprocal matrix of $f_{\alpha\beta}$ corresponding to the term $f_{\lambda\mu}$, and

$$[\alpha\beta, \mu] = \frac{1}{2}\left(\frac{\partial f_{\alpha\mu}}{\partial x^{\beta}} + \frac{\partial f_{\beta\mu}}{\partial x^{\alpha}} - \frac{\partial f_{\alpha\beta}}{\partial x^{\mu}}\right).$$

The method of obtaining the solution of (12) in the form (13) is formally exactly the same as that used in obtaining the equations of the geodesics in a Riemann space.*

It may be remarked, in terms of the concepts of *parallelism* and *curvature*, as developed in the author's thesis mentioned in § 1, that along a geodesic the tangent to the geodesic remains parallel to itself, or in other words that a geodesic is a curve of constant direction. A geodesic is also characterized as a curve of zero first curvature.

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* See F. D. Murnaghan, *Vector Analysis and the Theory of Relativity*, 1922, pp. 89-90.

INTEGRO-DIFFERENTIAL EQUATIONS*

BY I. A. BARNETT

1. *Introduction.* It is proposed in this note to prove that if, in the integro-differential equation

$$\frac{\partial \varphi(x,t)}{\partial t} = \int_0^1 \varkappa(x,y)\varphi(y,t)\,dy, \tag{1}$$

the real and continuous kernel $\varkappa(x,y)$ is also symmetric or skew-symmetric then there can be no solutions of the form

$$e^{\lambda t}\sum_{k=0}^{p}\alpha_k(x)t^k \tag{2}$$

unless all $\alpha_k(x)$, $k \geqq 1$ are identically zero.† It will then be pointed out that a similar property holds for systems of linear differential equations

$$\frac{d\varphi_i}{dt} = \sum_{j=1}^{n}\varkappa_{ij}\varphi_j, \tag{3}$$

viz., if the $\varkappa_{ij}$ are real and constant (with respect to t), then in case the coefficient system is either symmetric or skew-symmetric, no real set of solutions can contain polynomials in t as factors. This fact seems, to the best of the writer's knowledge, to have escaped attention in the literature.‡

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* Presented to the Society, December 29, 1923.

† See I. A. Barnett, *Integro-differential equations with constant limits of integration*, this BULLETIN, vol. 26, pp. 193–203.

‡ I have recently had the privilege of examining, through the courtesy of Professor Brand of the University of Cincinnati, a course of lectures on linear differential equations that Professor Bôcher gave, in which the result for the symmetric case is given without proof. Professor Bôcher states there that Weierstrass worked out this case in 1858, ten years before he introduced elementary divisors. The writer has examined the collected works of Weierstrass but he was unable to find this result.

2. *Kernel Symmetric.* Consider first the integro-differential equation (2) where $\varkappa(x, y) = \varkappa(y, x)$. Suppose there were a solution of the form (2). Substituting (2) in (1), we find

$$\sum_{k=0}^{p-1} t^k \int_0^1 \varkappa(x, y)\alpha_k(y)dy + t^p \int_0^1 \varkappa(x, y)\alpha_p(y)dy = \sum_{k=0}^{p-1}(k+1)\alpha_{k+1}(x)t^k + \sum_{k=0}^{p-1}\lambda\alpha_k(x)t^k + \lambda\alpha_p(x)t^p.$$

Writing $\mu = 1/\lambda$ one sees that this is equivalent to the following simultaneous system of integral equations

$$(4)\qquad \begin{cases} \alpha_p(x) = \mu\displaystyle\int_0^1 \varkappa(x, y)\alpha_p(y)dy, \\ \alpha_k(x) = \mu\displaystyle\int_0^1 \varkappa(x, y)\alpha_k(y)dy - \mu(k+1)\alpha_{k+1}(x), \\ \qquad\qquad (k = p-1, p-2, \ldots, 2, 1, 0). \end{cases}$$

Suppose the integral equation

$$\alpha_p(x) = \mu\int_0^1 \varkappa(x, y)\alpha_p(y)dy$$

has a characteristic number μ_1 of index q_1 with the corresponding characteristic functions $\Phi_1^{(p)}, \Phi_2^{(p)}, \cdots, \Phi_{q_1}^{(p)}$ which may be supposed normed and orthogonalized. Then the solution $\alpha_p(x)$ is expressible as a linear combination of these solutions. Consider now the second equation in (4),

$$(5)\qquad \alpha_{p-1}(x) = \mu\int_0^1 \varkappa(x, y)\alpha_{p-1}(y)dy - \mu_p\alpha_p(x),$$

for the same characteristic number μ_1. Necessary conditions that this last non-homogeneous integral equation have solutions are that all the associated characteristic functions of the kernel $\varkappa(x, y)$ be orthogonal to the function $\alpha_p(x)$. But, since the kernel is, by hypothesis, symmetric, it follows that the set of associated characteristic functions is the same as the set of characteristic functions of $\varkappa$; viz., the set $\Phi_1^{(p)}, \cdots, \Phi_{q_1}^{(p)}$. This is impossible unless $\alpha_p(x) \equiv 0$ since $\int_0^1 \Phi_i^{(p)}(x)\Phi_i^{(p)}(x)dx \neq 0$, $i = 1, 2, \cdots, q_1$. Reasoning in the same way, one can prove that $\alpha_{p-1} \equiv \cdots \equiv \alpha_1 \equiv 0$.

3. *Kernel Skew-Symmetric.* Suppose now that the kernel $\varkappa(x, y)$ is skew-symmetric, i. e., $\varkappa(x,y) = -\varkappa(y,x)$. Equations (4) also hold in this case. It is well known in this case that the characteristic numbers are all of the form λi and that if $(\lambda_1 i, \Phi_1(x))$ are a characteristic number and function, respectively, of the kernel $\varkappa(x,y)$ then $\overline{\Phi}_1(x)$ is the associated characteristic function for the same characteristic number $\lambda_1 i$ where $\overline{\Phi}_1(x)$ is the conjugate of $\Phi_1(x)$. Hence, necessary conditions for the existence of a solution of equation (5) would be $\int_0^1 \alpha_p(x)\overline{\Phi}_j(x)dx = 0$, $j = 1, \cdots, q_1$. If then the q_1 characteristic functions $\Phi_1^{(p)}, \cdots, \Phi_{q_1}^{(p)}$ and the q_1 associated characteristic functions $\overline{\Phi}_1^{(p)}, \cdots, \overline{\Phi}_{q_1}^{(p)}$ are supposed normed and biorthogonalized, it is clear that not all of these necessary conditions can be satisfied since at least one of them reduces to $\int_0^1 \Phi_j^{(p)}(x)\overline{\Phi}_j^{(p)}(x)dx$ which equals unity.

4. *Analogy for Differential Equations.* Consider now the system of differential equations

$$\frac{d\varphi_i}{dt} = \sum_{j=1}^{n} \varkappa_{ij}\varphi_j, \tag{3}$$

and a set of solutions $\{\varphi_1(t), \cdots, \varphi_n(t)\}$. It is well known that if $\{\psi_1(t), \cdots, \psi_n(t)\}$ is any set of solutions of the system of differential equations adjoint to (3), one has the relation

$$\varphi_1(t)\psi_1(t) + \cdots + \varphi_n(t)\psi_n(t) = \text{constant.}$$

But in case the coefficient system $\varkappa_{ij}$ is skew-symmetric, the set $\{\psi_i(t)\}$ coincides with the set $\{\varphi_i(t)\}$* and consequently one must have $\varphi_1^2(t) + \cdots + \varphi_n^2(t) = \text{const.}$ In other words the set $\{\varphi_i(t)\}$ cannot contain polynomials in t as factors. In the symmetric case a solution of a given system with t replaced by $-t$ is a solution of the adjoint system and so again we have the same result.†

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* Goursat-Hedrick, *Mathematical Analysis*, vol. 2, part. II, p. 157.

† The writer is indebted to Professor O. D. Kellogg for suggesting this proof by the method of adjoint systems.

A NEW TYPE OF DOUBLE SEXTETTE CLOSED UNDER A BINARY (3,3) CORRESPONDENCE*

BY LOUISE D. CUMMINGS

1. *Introduction.* In connection with the Poncelet theorem that if a polygon of n sides can be inscribed in one conic and circumscribed to a second conic then an infinite number of these polygons exist, much investigation, both by elliptic functions and by algebraic methods, has been effected concerning the closure property for the (2,2) correspondence of the double binary forms. For the (3,3) correspondence the direct algebraic attack upon even the point sets of low orders has been, until rather recently, somewhat neglected.

Franz Meyer, among others, studied the (3,3) correspondence of four points and four planes and obtained the surprising result that if there is a first tetrahedron inscribed in one cubic curve and circumscribed to a second cubic curve there may not be a second tetrahedron, but if there is a second tetrahedron then an infinity of these tetrahedrons occur. The existence of one particular closed set of seven points and seven planes in a (3,3) correspondence, with the poristic property like the Poncelet polygons, was established in 1915 by White† and confirmed by Coble‡ who has investigated the general (m, n) correspondence. While Coble has not attempted an exhaustive classification, he has listed fourteen types, old and new, which are poristic configurations of double binary forms, among them one closed set of five points in a (3,3) correspondence. For periodic sets of six points in a (3,3) correspondence particular results are lacking. In this investigation of the (3,3) correspondence for $n = 6$, two non-congruent types of sextettes have been discovered by

* Presented to the Society, October 25, 1924.

† PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 1 (1915), p. 464.

‡ AMERICAN JOURNAL, vol. 43, No. 1 (Jan., 1921).

a simple algebraic method which is applicable to higher values of n.

A polyhedron of n faces, including the tetrahedron as the special case for $n = 4$, is suggested as a possible geometrical interpretation of the (3,3) correspondence for all values of n.

2. *The Special Problem.* The (3,3) correspondence

$$(a_0x^3+b_0x^2+c_0x+d_0)y^3+(a_1x^3+b_1x^2+c_1x+d_1)y^2+$$
$$(a_2x^3+b_2x^2+c_2x+d_2)y+(a_3x^3+b_3x^2+c_3x+d_3) = 0$$

contains fifteen constants. If the binaries x and y vary from 0 to ∞ and are independent projectively, this gives in general a non-closed correspondence. In an earlier paper* the author established the existence of one type of closed double sextette in a special (3,3) correspondence, possessing the poristic property like the Poncelet polygons in the (2,2) correspondence. This paper exhibits the nature of a second type of poristic sextette, and shows that only two non-congruent types exist. Two related gauche cubic curves are used for exemplification in order that this investigation may be compared readily with Meyer's study of the double quartette in which some of the generalized theorems appear to be erroneous. The points x lying on a twisted cubic K are represented by the six parameters a, b, c, d, e, f; the planes y osculating a second twisted cubic K' are represented by the parameters 1, 2, 3, 4, 5, 6 used as symbols; the double sextette is to be subject to the cyclic substitution $S \equiv (a\,b\,c\,d\,e\,f)(1\,2\,3\,4\,5\,6)$ and the correspondence is to be invariant under S and its powers. The relations between the points x and the planes y are shown in the two following mutual arrays I and I′, where the point $x = a$ lies in the triad of planes 1, 2, 4 or the plane $y = 1$ passes through the triad of points a, d, f.

* To appear in PROCEEDINGS INTERNATIONAL MATHEMATICAL CONGRESS, Toronto, 1924.

I

x	a	b	c	d	e	f
y	1	2	3	4	5	6
	2	3	4	5	6	1
	4	5	6	1	2	3

I'

y	1	2	3	4	5	6
x	a	b	c	d	e	f
	d	e	f	a	b	c
	f	a	b	c	d	e

3. *Determination of a Normal Form.* For the (3,3) correspondence with a closed cycle of six points, investigation shows that the form with fifteen constants may, without loss of generality, be replaced by a determinantal form which by the aid of the binary identities is reducible to an expression in four terms. Therefore the following trial form is set up initially,

$$F(x,y) \equiv A\cdot xb\cdot xc\cdot xd\cdot y1\cdot y2\cdot y4 + B\cdot xc\cdot xd\cdot xa\cdot y2\cdot y3\cdot y5 + C\cdot xd\cdot xa\cdot xb\cdot y3\cdot y4\cdot y6 + D\cdot xa\cdot xb\cdot xc\cdot y4\cdot y5\cdot y1 = 0,$$

where for brevity xa represents the determinant of order 2 or the simple difference $x-a$.

This form visibly satisfies twelve of the conditions required by the array I, all on a, b, c, d; the application of the remaining six conditions furnished by the pairs $x = e$, $y = 5, 6, 2$ and $x = f$, $y = 6, 1, 3$ gives six equations linear in A, B, C, D such as

$$A\cdot eb\cdot ec\cdot ed\cdot 61\cdot 62\cdot 64 + B\cdot ec\cdot ed\cdot ea\cdot 62\cdot 63\cdot 65 + D\cdot ea\cdot eb\cdot ec\cdot 64\cdot 65\cdot 61 = 0.$$

Three of these equations determine the constants B/A, C/A, D/A, and with the binary identities allow the reduction of the form to any desired normal form such as the following:

$$\begin{aligned} F(x,y) \equiv\ & 12\cdot 35\cdot 56\cdot ae\cdot cf\cdot xb\cdot xc\cdot xd\cdot y1\cdot y2\cdot y4 \\ & + 61\cdot 25\cdot 14\cdot bf\cdot ce\cdot xa\cdot xc\cdot xd\cdot y2\cdot y3\cdot y5 \\ & + 12\cdot 15\cdot 25\cdot ce\cdot cf\cdot xa\cdot xb\cdot xd\cdot y3\cdot y4\cdot y6 \\ & + 15\cdot 23\cdot 26\cdot cf\cdot de\cdot xa\cdot xb\cdot xc\cdot y4\cdot y5\cdot y1 = 0. \end{aligned}$$

The three remaining equations, which might conceivably impose redundant conditions upon the twelve parameters, give for the special case of the double sextette three in-

dependent relations E_1, E_2, E_3, equivalent to an invariant set of fifteen, where

$$E_1 \equiv \frac{ab\cdot cd}{ac\cdot bd} - \frac{23\cdot 45}{24\cdot 35} = 0; \quad E_2 \equiv \frac{ab\cdot ce}{ac\cdot be} - \frac{23\cdot 46}{24\cdot 36} = 0;$$

$$E_3 \equiv \frac{ab\cdot cf}{ac\cdot bf} - \frac{23\cdot 41}{24\cdot 31} = 0.$$

These fifteen invariant relations $E_i = 0$, etc. express a unique projectivity of the x range to the y range, namely, that

$$\begin{matrix} a & b & c & d & e & f \\ 2 & 3 & 4 & 5 & 6 & 1. \end{matrix}$$

is projective to

The form $F(x, y)$ thus admits by its construction one configuration of period 6; is this an isolated closed sextette, or does the form admit an infinitude of these configurations and so possess the poristic property?

4. *Conditions for the Porism of the Form.* To ascertain if the sextette is necessarily poristic, the twelve parameters which express the correspondence are replaced by the neighboring set $a+da$, $b+db$, $\cdots$, $6+d6$, which must determine the same correspondence by the same formal equation. That is, the first differential must be a doubly cubic form identical with the original form $F(x, y)$, so that

$$F(x, y) + \frac{\partial F}{\partial a} da + \cdots + \frac{\partial F}{\partial 6} d6 \equiv M \cdot F(x, y).$$

A comparison of the coefficients of these two forms may be established by making use again of the eighteen pairs of points (x, y) given by the array I. These furnish eighteen equations of condition whose explicit form is derivable either by an algebraic process or by the following simple geometric method, in which for brevity da is replaced by a'. The array I′ shows that the form $F(x, y)$ regarded as a plane sextic is cut by the line $y = 1$ in the three points a, d, f and in a triple point at infinity, while the neighboring line $y = 1+1'$ cuts the curve in the points

$a+a'$, $d+d'$, $f+f'$ and in the same triple point at infinity. The slopes at the points $(a, 1)$, $(d, 1)$, $(f, 1)$ furnish the three equations

$$1' = \left(\frac{dy}{dx}\right)_{a,1} a' = \left(\frac{dy}{dx}\right)_{d,1} d' = \left(\frac{dy}{dx}\right)_{f,1} f'.$$

Replacing $\left(\frac{dy}{dx}\right)_{a,1}$ by $\left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right)_{a,1}$ and making similar substitutions for the slopes at $(d, 1)$ and $(f, 1)$, we have the three equations

$$\left(\frac{\partial F}{\partial x}\right)_{a,1} a' + \left(\frac{\partial F}{\partial y}\right)_{a,1} 1' = 0; \quad \left(\frac{\partial F}{\partial x}\right)_{d,1} d' + \left(\frac{\partial F}{\partial 1}\right)_{d,1} 1' = 0;$$

$$\left(\frac{\partial F}{\partial x}\right)_{f,1} f' + \left(\frac{\partial F}{\partial y}\right)_{f,1} 1' = 0.$$

The substitution S applied to each of these three equations gives three cycles of six equations or eighteen equations of condition to be satisfied by the twelve differentials $a', b', \cdots, 5', 6'$. To establish the consistency of these eighteen equations and the uniqueness of the solution their number must be reduced to eleven. This can be done if these equations satisfy the two following identities.

(1) $\frac{1'}{2'} \cdot \frac{2'}{3'} \cdot \frac{3'}{1'} \equiv 1$; (2) $\frac{1'}{2'} \cdot \frac{2'}{3'} \cdot \frac{3'}{4'} \cdot \frac{4'}{5'} \cdot \frac{5'}{6'} \cdot \frac{6'}{1'} \equiv 1$,

where

$$\frac{1'}{2'} = \frac{13 \cdot 16 \cdot ad \cdot af \cdot bc \cdot ce}{23 \cdot 26 \cdot ae \cdot cf \cdot ab \cdot cd},$$

and the other ratios have values similar in form. Now both of these identities are found to be satisfied if

$$\frac{ac \cdot bd \cdot fe}{ae \cdot bc \cdot fd} = 1,$$

that is, if the three pairs of points f, c; a, d; b, e are in a quadric involution.

The identity (1) enables us, with the aid of the cyclic property, to discard six of the eighteen equations, while the identity (2) disposes of one more, leaving eleven linear equations for the determination of the eleven ratios

$$\frac{a'}{1'}, \frac{b'}{1'}, \cdots, \frac{6'}{1'}.$$

Therefore the eighteen equations become consistent and determine uniquely the ratios of the twelve differentials. This gives then the differential operator which generates in the Lie sense a singly infinite group of transformations of the curves K and K' into themselves, or, as we prefer to regard it, which transforms the sextette of points on the curve into a simple infinity of sextettes similarly related to the curves K and K'. Hence we have the following theorem.

THEOREM. *Six points selected arbitrarily on a twisted cubic K can form in a* (3,3) *correspondence an isolated closed system of the type here sought, but if the six points are specialized as three pairs in a quadric involution on the cubic then the sextette is poristic and slides along the cubic K while the six associated planes continue to osculate the same second curve K' of class* 3.

5. *The Secant-Axes of the Double Sextette.* In our closed set of six points and six planes the array I′ shows a line *ad* joining two points of K and lying in two planes 1 and 4 of K'—a line which is a secant of K and an axis of K'. So also two other secant-axes *be* and *cf* are obviously present. Now a well known theorem of Cremona fixes the number of secants of one twisted cubic, which are at the same times axes of another twisted cubic, at six, and while three of these secant-axes are in evidence the whereabouts of the remaining three requires further investigation.

6. *Meyer's Erroneous Statement concerning the Hurwitz Relation.* A study by this method of investigation of four

points and four planes in a (3,3) correspondence presents no difficulty and gives results in entire agreement with those obtained by Meyer for the double quartette. But a discrepancy arises from Meyer's further assertion that whenever the number of secant-axes is infinite, then the curves K and K' are necessarily in the Hurwitz relation, that is, these lines can be grouped in sets of six to form *the edges of infinitely many tetrahedrons*, all inscribed in K and circumscribed to K'. The Hurwitz relation exists for the double quartette but Meyer's assertion that it obtains in every poristic system is erroneous, as this double sextette shows. The secants through a are ad, ac, af and of these ad alone is a secant-axis. Now the Hurwitz relation requires at each point of the curve K three secant-axes, *whereas the sextette possesses one and no more*—other essential conditions also are not fulfilled.

This discrepancy in results can be harmonized only if for $F(x, y) = 0$, we find $A \equiv 0$, $B \equiv 0$, $C \equiv 0$, $D \equiv 0$, but as each of these coefficients is a single product of differences, this cannot occur unless points are assumed coincident.

7. *The Non-Congruent Types of Double Sextettes.* Since this correspondence is to be invariant under the cyclic substitution $S \equiv (abcdef)(123456)$ only three non-congruent types can occur, namely a with the triad 1, 3, 5; a with the triad 1, 2, 4; and a with the triad 1, 2, 3. The type a with 1, 3, 5 is not properly sextic and so is omitted; a with 1, 2, 4 is discussed in this paper. The double sextette in which a is associated with the planes 1, 2, 3 was investigated in an earlier paper* and was found to be poristic for six points specialized as elements in a quadric involution, precisely similar to the specialization required for the double sextette here discussed. In the type a; 1, 2, 3 the six secant-axes are all in evidence but the conditions essential for the Hurwitz relation are not fulfilled.

* Loc. cit.

8. *The Poristic Double Quintette.* This method of investigation has been applied to five points and five planes, and gives results in agreement with Coble's theorem concerning the existence of a double quintette in a (3,3) correspondence possessing the poristic property.

Under the substitution $S \equiv (a\,b\,c\,d\,e)(1\,2\,3\,4\,5)$ all possible types are congruent to the type a with the triad of planes 1, 2, 3. A normal form is

$$\begin{aligned} F(x, y) \equiv\ & 45 \cdot ae \cdot xb \cdot xc \cdot xd \cdot y1 \cdot y2 \cdot y3 \\ & + 51 \cdot be \cdot xc \cdot xd \cdot xa \cdot y2 \cdot y3 \cdot y4 \\ & + 12 \cdot ce \cdot xd \cdot xa \cdot xb \cdot y3 \cdot y4 \cdot y5 \\ & + 23 \cdot de \cdot xa \cdot xb \cdot xc \cdot y4 \cdot y5 \cdot y1 = 0. \end{aligned}$$

Conditions are imposed upon the ten differentials a', b', $\cdots$, $4'$, $5'$ by fifteen equations similar in form to those which occur for the double sextette. The fifteen equations are reducible to nine consistent equations, which determine uniquely the nine ratios

$$\frac{a'}{1'}, \frac{b'}{1'}, \cdots, \frac{5'}{1'},$$

if the set a b c d e is projective to 2 3 4 5 1. No other conditions being imposed upon the parameters, a correspondence of period 5 can be determined with five arbitrary points for a fundamental set. The quintette is poristic; five secant-axes are ab, bc, cd, de, ea but the secant-axes do not fulfil the Hurwitz relation, so that here again Meyer's assertion is found incorrect.

9. *A Geometric Interpretation.* Since the Hurwitz relation does not hold for $n = 5$ nor for $n = 6$, there is for these cases no tetrahedron which slides along the curve K while its faces continue to osculate the second cubic K', as in the case of $n = 4$. However, one geometric figure applicable to all cases may be visualized in the following manner. From a polyhedron with n—1 lateral faces, such that no four of these lateral faces pass through a common point, cut a wedge whose upper and lower

bases intersect one of the lateral faces in the same straight line. The n planes which form the faces of the solid intersect in

$$\frac{n(n-1)(n-2)}{6}$$

vertices, and n of these vertices slide along the curve K while the n planes which form the faces continue to osculate the curve K'. This solid obviously becomes the tetrahedron for $n = 4$.

10. *Conclusion.* The method here employed is applicable to n points and n planes in a (3,3) correspondence. The determination of a normal form disposes of fifteen of the conditions imposed by the n-by-3 array I, leaving $3(n-5)$ conditions to be imposed upon the $2n$ parameters. The difficulty of eliminating from these $3(n-5)$ conditions those which are redundant increases rapidly with the increase in n, due to the increasing complexity in the coefficients of the normal form. The $3n$ equations arising in the comparison of $F(x, y)$ and its first derivative can be reduced to $2n-1$ equations, just sufficient for the determination of the $(2n-1)$ ratios $\frac{a'}{1'}, \frac{b'}{1'}, \cdots$, by two test identities similar to (1) and (2) used above. While the increase in n is accompanied always by an increase in the number of determinants in $\frac{a'}{1'}, \cdots$, nevertheless a further exploration of the field, by this method, does not present insurmountable difficulties for point sets of low orders.

The exhibition of these poristic point sets is one concrete step towards the realization of a suggestion offered by Mr. White in a paper on Poncelet polygons—"That all Poncelet systems are associated with linear involutions upon rational curves and that in this feature, possibly, lies even more promise of generalizations and discoveries than in Jacobi's brilliant and beautiful depiction by the aid of periodic functions".

VASSAR COLLEGE

SHORTER NOTICES

An Elementary Treatise on Frequency Curves and their Application in the Analysis of Death Curves and Life Tables. By Arne Fisher. Translated from the Danish by E. A. Vigfusson. With an introduction by Raymond Pearl. American Edition. New York, Macmillan, 1922. xv + 244 pp.

This work is divided into two chapters. The first, "An Introduction to the Theory of Frequency Curves", is practically the same as Chapters XIV–XVIII of *Mathematical Theory of Probabilities*, by the same author. This chapter takes up the treatment of frequency curves associated with the names of Laplace, Poisson, Charlier, Thiele and Gram, and serves as the theoretical introduction to the second chapter entitled "The Human Death Curve." In this chapter the author presents a method of constructing mortality curves from death lists by age and cause without knowledge of the number exposed to risk. In the United States this method has been received with various degrees of enthusiasm—from that of the extremely eulogistic introduction to the book down to utter condemnation by several well known actuaries whose views carry much weight. It is an axiom in actuarial science, often repeated and emphasized, that a mortality curve cannot be constructed from death lists alone. Some criticisms seem to imply that Fisher had done just this thing and a man of straw is created and warmly pelted. But he does not claim to get a mortality curve from death lists alone. It is true that he does not have the exposed to risk, but he has a substitute in something biologists and physicians have told him about the frequency distribution of deaths according to age from various classes of causes.

Dispensing with actuarial technicalities, the problem in the language of the theory of probabilities is as follows:

A bag contains a large number of black and white balls. Each ball has a number stamped upon it which may be any integer from 0 to 105. The black balls are also stamped with one of the letters a, b, c, d, e, f, g, h. Balls are drawn from the bag one at a time and replaced. A record is kept of the numbers and letters on the black balls, but no attention is paid to the numbers on the white balls. Later it is found desirable to construct a table of d_x/l_x, where d_x is the number of black balls, marked with the number x, and l_x is the total number of observed balls so marked. It is of course impossible to make such a table from the black ball records. But there is available information about the distribution of the balls in other bags

or of balls in the pile from which bags are filled with justifies one in making an assumption as to the laws of the frequencies in the bag in question. A table constructed upon this hypothesis may be tested by experience and found correct. The whole matter depends upon the validity of the assumption.

The method of the book assumes a given list of deaths according to sex, age and cause, and to this is added the biological hypothesis:

"The frequency distribution of deaths according to age from certain groups of causes of death among the survivors in a mortality table tend to cluster around certain ages in such a manner that the frequency distribution can be represented by either a Laplacean-Charlier or a Poisson-Charlier frequency curve."

With this goes a classification of causes of death according to some seven or eight groups. The mortality curve is considered as a compound curve made up from component frequency curves corresponding to the groups. The characteristics of these components are supposed known a priori.

Upon this foundation, Fisher is able to build up a mortality table. In the book he goes through the actual calculation of several tables that turn out to be satisfactory. In the mind of the reviewer the book marks a step forward in actuarial science in that it points out a method of attack on problems connected with that great mass of data in which the exposed to risk is difficult or impossible to find. There is no claim made that the method supersedes the old conventional method depending upon the number exposed to risk and losses among such exposures. However, it is possible that further experience with the method by the author and by others will show limitations in its applications. Before it is accepted by the conservative actuary, he will require much more evidence in the nature of satisfactory tables computed by many actuaries.

A. R. CRATHORNE

Analytical Mechanics. Second edition. By Edwin H. Barton. London, Longmans, Green and Company, 1924. 593 pp.

The first edition of this book was published in 1911. The new material in the second edition consists mainly of alternative proofs and more examples. About three hundred examples chosen from London examination questions set in the years 1911–23 have been added, bringing the total number to more than a thousand. To these a complete set of answers is now given separately at the end.

The text, divided into six parts, covers the field of mechanics quite thoroughly. Besides a preliminary survey of the scope of mechanics, the introductory part contains a collection of formulas from algebra, geometry, trigonometry, and the calculus. The second part gives a

very complete account of the kinematics of a point and of a rigid body, including mechanisms and strains. The treatment of kinetics is preceded by a discussion of Newton's principles and criticisms of these principles by Mach and others. Statics follows kinetics and includes a good chapter on attractions and potential. Part V contains a chapter on hydrostatics and one on hydrokinetics. The sixth part consists of a short chapter on statics of elastic solids.

The author expects that most users of the book will bring to it some previous knowledge of the subject, and the elementary parts are briefly outlined to serve as a reference and for logical completeness. The general scope and treatment is designed to meet the needs of degree candidates of London and other universities. Very few, if any, American universities offer courses for which this book would be suitable as a text. But as a reference book for the teacher it would be very valuable because of the large number of examples and the very complete index.

W. R. LONGLEY

Clerk Maxwell's Electromagnetic Theory (The Rede Lecture for 1923). By H. A. Lorentz. Cambridge University Press, 1923. 35 pp.

The Theory of Relativity, Studies and Contributions. By Archibald Henderson, A. W. Hobbs, and J. L. Lasley. University of North Carolina Press, Chapel Hill, N. C., 1924. xiii+99 pp.

Some men can put into a brief exposition a revelation of such insight and appreciation of the scientific significance of their topics as they may have won after years of detailed study. Among authors who have written on the theory of relativity, A. S. Eddington has written expositions of the type which I have in mind, his Romanes Lecture (delivered in 1922 at the Sheldonian Theater in Oxford) being an excellent example. The more recent lecture by Lorentz, mentioned above, is in the same class, conveying a nontechnical appreciation of the place of Maxwell's theory in the modern development of physics.

On the other hand the booklet by Professor Henderson and his colleagues is a disappointment to those who have read Professor Henderson's magazine articles on Bernard Shaw. One might have expected a mathematician of comparatively broad interests to bring to the subject of relativity a more novel point of view and a more significant insight into the physical meaning of the subject. Instead he has hurriedly and carelessly written a formal outline of the theory. As an example of carelessness I will refer to page 66 where the reader is suddenly confronted with the statement that the mass of the sun is 1.47 kilometers, with no explanation of the use of a linear unit for mass. Eddington's Report to the Physical Society of London is a better and less expensive work of approximately the same scope.

C. N. REYNOLDS, JR.

Einleitung in die Mengenlehre. 2d edition. By A. Fraenkel. Berlin, Julius Springer, 1923. viii + 251 pp.

In this edition this book appears as volume 9 of the series, *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen.* The author in writing this edition tried to produce a book which would fulfill the general requirements of that series and also the purpose of the first edition*, i. e. to give an exposition of the elements of Mengenlehre which would be clearly comprehensible to one of little or no mathematical training. The prolixity of exposition necessary in realizing the latter aim hinders the realization of the former. The elegance and, in particular, the conciseness which the initiated desires is missing. However, the author seems to have come as near as possible to fulfilling with a single book both purposes.

The first 151 pages of the new edition contain almost verbatim all of the material of the first 129 pages of the old edition and in addition the introduction of certain concepts, more abstruse than those of the earlier book, and the details of certain proofs which previously were give only in outline. The material of chapter 12, pp. 129–151, of the old edition is elaborated in 90 pages, pp. 151–241, of the 2d edition. These pages are devoted to the foundations of the subject. About half of this space is taken up with a historical sketch of the critical examination of the theory, including an exposition of the paradoxes which gave rise to this critique and a brief description of the various procedures devised to give a logically consistent theory. Here are briefly explained and contrasted intuitionalism, which recently has been the subject of considerable work by Brouwer and Weyl, the method of logicizing as developed by Russell and Whitehead and J. Königs, and the axiomatic method as employed by Zermelo and Hilbert. The remaining half is devoted to a detailed exposition of the axiomatic developement of Mengenlehre which Zermelo produced in the period from 1904–1908 and to a sketch of Hilbert's recent work on the question of a proof of the consistency of a system of postulates. In connection with the Zermelo axiomatic the author gives in fine print an exposition of his contribution to the clarifying of the notion "eine definite Aussage" as used by Zermelo.

The exposition of the Zermelo-Hilbert way of setting up a theory of sets is well done but the brevity with which the other methods, logicization and intuitionalism, are presented makes it impossible to give an exposition of the latter methods on a par with that of the former. This is to be regretted. However, in these 90 pages we have the most readily available means of obtaining an acquaintance with the present state of the theory of the foundations of Mengenlehre

* See this BULLETIN, vol. 27 (1921), pp. 333–334.

and, indeed, of mathematics. Although none of the various theories, the logical theories of Russell and Whitehead and of J. Königs, the intuitional theory, suggested by Kronecker and developed by Brouwer and Weyl, the axiomatic theory brought forth by Zermelo and contributed to by Fraenkel and others and Hilbert's theory of proof, which are described here, is in a finished form, they all seem to lead to viewing mathematics and logic as a game in which the pieces and the rules of operation are defined as concretely as possible.

The book closes with a chapter of a few pages which contains a discussion of the connection between Mengenlehre and other branches of mathematics and a bibliography of the subject. These last pages are nearly identical with the corresponding ones of the first edition.

No errors in definitions or proofs have been noted, and the typography is excellent, as in previous volumes of the Springer series.

G. A. PFEIFFER

Theorie der Differentialgleichungen. Vorlesungen aus dem Gesamtgebiet der gewöhnlichen und der partiellen Differentialgleichungen. By Ludwig Bieberbach. Berlin, Julius Springer. 1923. viii + 317 pp.

"To write a text book on differential equations which shall be a suitable textbook, and at the same time set forth the spirit, methods and results of the theory in all of their aspects, so that the student shall be prepared to read original papers intelligently—that appears to be an impossibility." With these words Bieberbach opens the preface to the book under review, which is intended only as an introduction to the subject. Its scope is best brought out by giving the titles of the four sections into which the book is divided.

Section I. Ordinary differential equations of the first order.

Section II. Ordinary differential equations of the second order.

Section III. Partial differential equations of the first order.

Section IV. Partial differential equations of the second order.

It is limited therefore to equations of the first and second order.

The author is much more interested in bringing out the meaning and the interpretation of differential equations than he is in setting exercises for the student; in fact, there are no exercises at all. Existence proofs are given, and discussions of the nature of the integral curves. He does not limit himself to real values of the variables, for in each of the first two sections he has a chapter dealing with the the complex values of the variable and the nature of singularities.

While the book is elementary in character, its spirit is scholarly, and the book is well worth while to the student who has not gone deeply into the subject of differential equations.

W. D. MACMILLAN

NOTES

The opening number of volume 47 of the AMERICAN JOURNAL OF MATHEMATICS (January, 1925) contains the following papers: *L'expression la plus générale de la "distance" sur une droite*, by Maurice Fréchet; *Second paper on tensor analysis*, by G. Y. Rainich; *Generalization of certain theorems of Bohl*, by F. H. Murray; *Expansion problems in connection with the hypergeometric differential equation*, by B. P. Reinsch.

The third number of volume 25, series 2, of the ANNALS OF MATHEMATICS (March, 1924) contains: *On the method of least mth powers for a set of simultaneous equations*, by Dunham Jackson; *The derivative of the general integral*, by P. J. Daniell; *On a class of transformations in function space*, by I. A. Barnett; *Congruences of circles studied with reference to the surface of centers*, by J. M. Thomas; *On Neumann's existence proof*, by W. F. Osgood; *On the automorphic functions of the group* (0,3; 2, 4, 6), by H. H. Dalaker; *Note on Dirichlet's series with complex exponents*, by Einar Hille; *On the types of monoidal involutions*, by Virgil Snyder; *An inequality for the roots of an algebraic equation*, by J. L. Walsh. The fourth number (June, 1924) contains: *Differential equations from the group standpoint*, by L. E. Dickson.

At the annual meeting of the Mathematical Association of America in Washington, the following officers were elected: Professor J. L. Coolidge, president; Professors A. A. Bennett and Dunham Jackson, vice-presidents; Professors R. C. Archibald, L. P. Eisenhart, E. V. Huntington, and H. L. Rietz, members of the Board of Trustees.

Among the officers of the American Association for the Advancement of Science elected at the Annual Meeting of 1924, held at Washington, are the following: President, Professor M. I. Pupin, of Columbia University; vice-president, Section A (mathematics), Professor W. H. Roever, of Washington University; secretary, Section A, Professor R. C. Archibald; member of the executive committee, Professor E. B. Wilson, Harvard School of Public Health. Professors Arnold Dresden and E. B. Stouffer have been appointed representatives of the American Mathematical Society on the Council of the Association for 1925. The next Annual Meeting will be held at Kansas City.

The following have been elected officers of the London Mathematical Society: President, A. L. Dixon; vice-presidents. L. N. G. Filon, H. Hilton, and W. H. Young; treasurer, A. E. Western; secretaries, G. H. Hardy and G. N. Watson.

The next meeting of the International Research Council will be held at Brussels, beginning July 7, 1925. The International Physics Union will also meet in Brussels in July. The International Astronomical Union will meet at Cambridge, England, July 14–22.

A journal devoted largely to mathematics has recently been founded at Kief, the BULLETIN DE LA CLASSE DES SCIENCES PHYSIQUES ET MATHÉMATIQUES DE L'ACADÉMIE DES SCIENCES DE L'OUKRAINE; Professor N. M. Kryloff is editor. The Academy, which was founded in November, 1918, has also begun publication of its MÉMOIRES, CLASSE DES SCIENCES.

The Hungarian Francis-Joseph University of Szeged (formerly in Kolozsvar, Transylvania) has recently founded a periodical of general character, the ACTA LITERARUM AC SCIENTIARUM, which publishes articles in English, French, and German. In the first volume, the section devoted to mathematics contains articles by F. Riesz, Kürschák, Fejér, Haar, M. Riesz, Kerékjárto, and others. This section will form annually a volume of about 250 pages, containing original articles and also some notes on bibliography. The subscription price is three dollars per volume. Subscriptions should be sent to the Editor, Professor F. Riesz, Dugonicster 13, Szeged, Hungary.

A new series of monographs on mathematical topics has been established, under the patronage of the Société Mathématique de France and of the scientific academies of Paris, Servia, Belgium, Bucarest, Poland, Ukraine, Spain, Prague, Rome (dei Lincei), and Sweden, and will be known as MÉMORIAL DES SCIENCES MATHÉMATIQUES. It will be issued by the firm of Gauthier-Villars, under the general direction of H. Villat, editor of LIOUVILLE'S JOURNAL. The first of these monographs, by P. Appell, entitled *Sur une Forme Générale des Équations de la Dynamique*, has just appeared. Six others are in press, and no less than eighty-nine others are in preparation. The purpose is to present in very brief form the fundamental ideas in an extensive range of mathematical topics.

There has been established in Moscow a Joint Information Bureau, whose purpose is to improve the contact between scientists of Russia and those of other countries, and to facilitate an exchange of scientific periodical and non-periodical publications. Personal exchange of their works between scientists may be arranged, as may also the publication of articles. Those interested should address the Russian Information Bureau, 2819 Connecticut Ave., N. W., Washington, D. C.

Cambridge University has awarded its John Bernard Seely prize in aeronautics to W. A. Johnson, of Peterhouse, for an essay on *Dynamical similarity and scale effects.*

The Institut de France has awarded its Osiris prize to Professor Charles Fabry, for his interferometer.

The Paris Academy of Sciences announces the following awards: the Poncelet prize to Professor Ernest Vessiot, of the University of Paris, for his mathematical work; the Francœur prize to Commandant Ernest Malo, for his researches in algebra and arithmetic; the Fourneyron prize (founded in honor of Benoit Fourneyron, inventor of the turbine) to Marcel Crozet-Fourneyron, for his inventions of apparatus for regulating hydraulic motors and for his history of the invention of the turbine; the Henri de Parville prize to Commandant Paul Bloch, for his *Essai de balistique aérienne*; the La Caze prize to Professor Paul Langevin, of the Collège de France, for his scientific work; the Hébert prize to Professor Edgar Haudié, of the Ecole Navale, for his *Cours d'Electricité Générale*; the grand prix des sciences mathématiques to Professor Paul Montel, of the University of Paris, for his work on sequences of analytic functions; the Vaillant prize to Professor Claude Guichard, of the University of Paris, for his work in geometry; a prize from the Hirn foundation to Professor Georges Giraud, of Clermont-Ferrand, for his scientific work; a prize from the Henri Becquerel foundation to Professor René Garnier, for his work in differential equations.

Professor G. Fubini, of the Turin Technical School, has presented a sum of money to the Unione Matematica Italiana to found a prize to be called the "Premio ing. Lazzaro Fubini"; the prize is to be awarded every two years, under conditions to be announced by the Unione, for some unpublished work by a young Italian mathematician.

It is announced that the second award of the Cesare Arzelà prize of the Bologna Academy of Sciences will be made in December, 1928; this prize is awarded to a doctor of an Italian University for work in the theory of functions of real variables completed within five years of his graduation.

Professor Emile Picard, permanent secretary of the Paris Academy of Sciences, has been elected a member of the Académie Française.

Professors L. Bianchi, of the University of Pisa, and E. Goursat, of the University of Paris, have been elected associates in the section of mathematical and physical sciences of the Royal Academy of Belgium.

Professors N. Bohr, of the University of Copenhagen, and M. von Laue, of the University of Berlin, have been elected foreign members of the Royal Academy dei Lincei of Rome.

Professor A. W. Bickerton, of London, has been elected an honorary member of the New Zealand Astronomical Society in recognition of his work on cosmic evolution.

Professor Harry Bateman, of the California Institute of Technology, has been elected a member of the American Philosophical Society.

At the University of Lille, Dr. H. Beghin has been promoted to the chair of rational mechanics.

Professor Elie Cartan, of the University of Paris, has been transferred from the professorship of rational mechanics to that of higher geometry.

Professor C. Riquier, of the University of Caen, has been designated honorary professor.

At the University of Florence, which has been established recently, Professor E. Ciani, of the University of Genoa, has been appointed professor of higher geometry.

At the University of Milan, which has been established recently, Professors L. Berzolari and G. Vivanti, of the University of Pavia, and G. A. Maggi, of the University of Pisa, have accepted the professorships of higher geometry, higher analysis, and mathematical physics, respectively.

At the University of Naples, Professor M. Picone, of the University of Pisa, has been appointed to the chair of calculus, as successor to Professor G. Torelli, who has retired from active teaching; Professor G. Sannia, of the University of Modena, has been transferred to the chair of descriptive geometry.

At the University of Padua, Professor G. Vitali, of the University of Modena, has been appointed professor of calculus as successor to Professor F. d'Arcais, who has retired from active teaching.

Professor A. Palatini, of the University of Parma, has been appointed professor of rational mechanics.

At the University of Pisa, Professor E. Daniele, of the University of Modena, and Professor O. Lazzarino, of the University of Catania, have been appointed to professorships of rational and celestial mechanics.

The following appointments as privat-docents in Italian Universities are announced: University of Bologna, Dr. G. Belardinelli; University of Turin, Dr. U. Cassina; University of Naples, Dr. A. Finzi; University of Florence, Dr. G. Sansone.

The title of emeritus professor of mathematics in the University of London has been conferred on Dr. M. J. M. Hill, who held the chair of pure mathematics at University College from 1884 to 1923.

Professor E. T. Whittaker has been elected dean of the faculty of arts of the University of Edinburgh.

Professor L. P. Eisenhart has been elected dean of the faculty of Princeton University.

Dr. R. E. Gleason, of Princeton University, has been appointed professor of mathematics at Temple University, Philadelphia.

Dr. P. E. Heinke, of the United States Naval Academy, has accepted a position as mathematical analyst at Langley Field, Virginia.

At the University of Vermont, Mr. F. W. Householder has been promoted to an assistant professorship.

Dr. L. M. Kells and Mr. W. A. Conrad, of the United States Naval Academy, have been promoted to assistant professorships of mathematics.

Associate Professor Arthur Kiernan, of the United States Naval Academy, has accepted a position with Ginn and Company.

Professor Solomon Lefschetz, of the University of Kansas, has been appointed to an associate professorship of mathematics at Princeton University.

Mr. W. E. Loring, of the University of Maine, has been appointed assistant professor of mathematics at Colby College, to fill the vacancy caused by the illness of Professor B. E. Carter.

Professor R. G. D. Richardson, of Brown University, secretary of the American Mathematical Society, has been granted leave of absence for the second semester of 1924–25.

At the Agricultural and Mechanical College of Texas, Assistant Professors P. K. Smith and F. W. Sparks have resigned, and Dr. W. P. Udinski has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
University of Vermont, Mr. H. A. Giddings;
Agricultural and Mechanical College of Texas, Messrs. Frank Ayres, A. A. Blumberg, and W. P. Stevens.

Professor E. A. Shaw, for twenty-five years professor of mathematics at Norwich University, has retired from active teaching.

Professor Hugo von Seeliger, of the University of Munich, died December 2, 1924, at the age of seventy-five.

Professor A. G. Hall, registrar of the University of Michigan, died January 11, 1925. Professor Hall had been a member of the American Mathematical Society since 1902.

Mr. D. H. MacPherson, instructor in mathematics at the Brooklyn Polytechnic Institute, died December 28, 1924, at the age of 22.

Mr. W. L. Putnam, attorney at law, of Boston, died July 26, 1924. Mr. Putnam had been a member of the American Mathematical Society since 1910, and had frequently been of great service to the Society in legal matters.

The death is announced of Dr. E. W. Stanton, formerly professor of mathematics at Iowa State College, at the age of seventy-four.

Dr. Brandreth Symonds, chief medical director of the Mutual Life Insurance Company, died August 10, 1924; Dr. Symonds was a member of the American Mathematical Society.

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APPELL (P.). Souvenirs d'un Alsacien. Paris, Payot, 1923.

CUNNINGHAM (A. J. C.). Binomial factorisations, giving congruence tables and factorisation tables. Volume 2, 4 + 79 + 215 pp. Volume 6 (supplement to volume 2), 3 + 103 pp. London, F. Hodgson, 1924.

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JUNG (H. W. E.). Algebraische Flächen. Hannover, Helwingsche Verlagsbuchhandlung, 1924. 16 + 420 pp.

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PICARD (E.). Pascal mathématicien. Paris, Gauthier-Villars, 1924. 23 pp.

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SCHUH (F.). Lessen over de hoogere algebra. Tweede deel. Groningen, Noordhoff, 1924. 358 pp.

ZORETTI (L.). See ROSENTHAL (A.).

PART II. APPLIED MATHEMATICS

BECQUELER (J.). Cours de physique à l'usage des élèves de l'enseignement supérieur et des ingénieurs. Tome 1: Thermodynamique. Paris, Hermann, 1924. 8+431 pp.

BEHM (H.). Kubiktabelle. Zur Bestimmung des Inhaltes von Rundhölzern nach Kubikmetern und Hundertteilern des Kubikmeters, mit angehängten Reduktionstafeln. Berlin, Springer, 1924. 72 pp.

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CLEMENTI (I.). I contattori elettrici. Teoria e pratica. Milano, Hoepli, 1924. 16+291 pp.

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——. Relativitätstheorie in mathematischer Behandlung. Autorisierte Übersetzung von A. Ostrowski und H. Schmidt. Mit Zusätzen und Ergänzungen und mit einem Anhang von A. Einstein. (Die Grundlehren der mathematischen Wissenschaften, Band 18.) Berlin, Springer, 1924.

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